Name: ID:

Math 317 Q1 Winter 2017 Quiz 5 Solutions

Mar. 24, 2017, 25 minutes

- The quiz has 3 problems. Total 10 + 1 points.

Question 1. (5 pts) *Prove or disprove: the infinite series $\sum_{n=0}^{\infty} \frac{e^{nx}}{n^2}$ converges uniformly on $(-\infty, 0)$.*

Proof. We prove the claim by Weierstrass's M-test. Let $a_n := \frac{1}{n^2}$. Then we have

$$\sum_{n=0}^{\infty} a_n \text{ converges}, \qquad \text{and } \left| \frac{e^{nx}}{n^2} \right| \leqslant a_n \text{ for all } x \in (-\infty, 0). \tag{1}$$

The conclusion now follows. □

Question 2. (5 pts) *Prove that $f(x) := \sum_{n=0}^{\infty} n^2 \, e^{-nx}$ is continuous on $(0, \infty)$.*

Proof. Let $x_0 \in (0, \infty)$ be arbitrary. Let $a := \frac{x_0}{2}, b := 2\, x_0$. We prove that $f(x)$ is uniformly convergent on $[a, b]$. To see this we apply Weierstrass's M-test. Define

$$a_n := n^2 \, e^{-n x_0/2}. \tag{2}$$

By ratio test we see that

$$\limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \limsup_{n \to \infty} \left(\frac{n+1}{n} \right)^2 e^{-x_0/2} = e^{-x_0/2} < 1. \tag{3}$$

Therefore $\sum_{n=0}^{\infty} a_n$ converges. As $|n^2 \, e^{-nx}| \leqslant a_n$ for all $x \in \left[\frac{x_0}{2}, \, 2\, x_0\right]$, the conclusion follows. □

Question 3. (1 bonus pt) *Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^2+n}$ converges uniformly on $\mathbb{R}$.*

Proof. We apply the Dirichlet test. Take $a_n(x) := (-1)^{n-1}$ and $b_n(x) := \frac{1}{x^2+n}$. Then clearly

i. $\sum_{n=1}^{N} a_n(x)$ are uniformly bounded for every $N \in \mathbb{N}$;

ii. For each fixed $x \in \mathbb{R}$, $b_n(x)$ is decreasing;

iii. $b_n(x) \longrightarrow 0$ uniformly.

Thus Dirichlet's test applies and the conclusion follows. $\square$