Name: ID:

Math 317 Q1 Winter 2017 Quiz 4 Solutions

Mar. 10, 2017, 25 minutes

- The quiz has 3 problems. Total 10 + 1 points.

Question 1. (5 pts) *Find all $x \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} \frac{e^{nx}}{n^2}$ is convergent. Justify your claim.*

Solution. We claim that the series converges if and only if $x \leqslant 0$.

- $x < 0$. In this case we apply the ratio test:

$$\left| \frac{e^{(n+1)x} / (n+1)^2}{e^{nx} / n^2} \right| = \frac{n^2}{(n+1)^2} e^x \longrightarrow e^x < 1 \tag{1}$$

  as $n \longrightarrow \infty$.

- $x = 0$. In this case the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which we know is convergent.

- $x > 0$. In this case we apply the ratio test:

$$\left| \frac{e^{(n+1)x} / (n+1)^2}{e^{nx} / n^2} \right| = \frac{n^2}{(n+1)^2} e^x \longrightarrow e^x > 1 \tag{2}$$

as $n \longrightarrow \infty$.

QUESTION 2. (5 PTS) *Let a_n, $b_n > 0$ and assume $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ are convergent. Prove that $\sum_{n=0}^{\infty} c_n$ is convergent where $c_n = a_n b_n$.*

Solution. As $a_n, b_n > 0$, the convergence of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is equivalent to the existence of $M \in \mathbb{R}$ such that

$$\sum_{n=0}^{N} a_n, \sum_{n=0}^{N} b_n < M \tag{3}$$

for all $N \in \mathbb{N}$. Now it is clear that

$$\sum_{n=0}^{N} c_n < \left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N} b_n\right) < M^2 \tag{4}$$

for all $N \in \mathbb{N}$. Consequently $\sum_{n=0}^{\infty} c_n$ is convergent.

Alternative solution. As $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, there holds $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$. Consequently there is $n_0 > 0$ such that for all $n > n_0$,

$$|a_n|, |b_n| < 1. \tag{5}$$

Thus for such n,

$$|c_n| \leqslant \frac{a_n^2 + b_n^2}{2} < \frac{a_n + b_n}{2} \tag{6}$$

and the conclusion follows from comparison.

QUESTION 3. (1 BONUS PT) *Let* $a_n > 0$ *and assume* $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ *are convergent. Prove that* $\sum_{n=0}^{\infty} c_n$ *is convergent where* $c_n = a_n b_n$.

Proof. We prove that $\sum_{n=0}^{\infty} c_n$ is Cauchy. Let $\varepsilon > 0$ be arbitrary. As $\sum_{n=0}^{\infty} a_n$ is convergent, there is $n_1 > 0$ such that for all $n > m > n_1$,

$$|a_{m+1}| + \cdots + |a_n| = |a_{m+1} + \cdots + a_n| < \varepsilon. \tag{7}$$

On the other hand, as $\sum_{n=0}^{\infty} b_n$ is convergent, there is $n_2 > 0$ such that for all $n > m > n_2$,

$$|b_{m+1} + \cdots + b_n| < 1. \tag{8}$$

In particular, $|b_n| < 1$ for all $n > n_2$. Now take $N := \max\{n_1, n_2\}$. Then for every $n > m > N$, there holds

$$|c_{m+1} + \cdots + c_n| \leqslant |a_{m+1} b_{m+1}| + \cdots + |a_n b_n| \leqslant |a_{m+1}| + \cdots + |a_n| < \varepsilon. \tag{9}$$ □