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MATH 317 Q1 WINTER 2017 QUIZ 3 SOLUTIONS

Feb. 17, 2017, 25 minutes

- The quiz has 3 problems. Total 10 + 1 points.

QUESTION 1. (5 PTS) *Calculate the surface area of the torus*

$$((a+b\cos\theta)\cos\varphi, (a+b\cos\theta)\sin\varphi, b\sin\theta) \tag{1}$$

where $0<a<b$.

Solution. We have

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^{2\pi} \|\sigma_\theta\times\sigma_\varphi\|\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \int_0^{2\pi}\int_0^{2\pi} b\,(a+b\cos\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= 4\,\pi^2\,a\,b. \end{aligned} \tag{2}$$

QUESTION 2. (5 PTS) *Let* $\boldsymbol{f}(x,y,z) := \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ *and* S *be the portion of* $x^2+y^2=z$ *with* $1\leqslant z\leqslant 4$*, oriented so that the normal points upward. Calculate* $\int_S \boldsymbol{f}\cdot\mathrm{d}S$

Solution. We parametrize S as

$$x=u, y=v, z=u^2+v^2, \qquad (u,v)\in D := \{1\leqslant u^2+v^2\leqslant 4\}. \tag{3}$$

Then

$$\boldsymbol{r}_u = \begin{pmatrix} 1 \\ 0 \\ 2\,u \end{pmatrix}, \boldsymbol{r}_v = \begin{pmatrix} 0 \\ 1 \\ 2\,v \end{pmatrix} \Longrightarrow \boldsymbol{r}_u\times\boldsymbol{r}_v = \begin{pmatrix} -2\,u \\ -2\,v \\ 1 \end{pmatrix}. \tag{4}$$

As the 3rd component is $1>0$, $\boldsymbol{r}_u\times\boldsymbol{r}_v$ points upward. Thus we have

$$\begin{aligned} \int_S \boldsymbol{f}\cdot\mathrm{d}S &= \int_D \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -2\,u \\ -2\,v \\ 1 \end{pmatrix}\mathrm{d}(u,v) \\ &= \int_D [-6\,u-4\,v+1]\,\mathrm{d}(u,v) \\ \int [-6\,u-4\,v]=0 \text{ due to symmetry} &= \int_D \mathrm{d}(u,v) = 3\,\pi. \end{aligned} \tag{5}$$

QUESTION 3. (1 BONUS PT) *Let $\Omega \subset \mathbb{R}^3$ be the region bounded by $x=0, y=0$, $z=0$ and $x+y+z=1$. Let $\partial\Omega$ be its boundary. Let $f, g, h \in C^1$. Prove*

$$\int_{\partial\Omega} f\,\mathrm{d}y \wedge \mathrm{d}z + g\,\mathrm{d}z \wedge \mathrm{d}x + h\,\mathrm{d}x \wedge \mathrm{d}y = \int_{\Omega} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{6}$$

where the normal of $\partial\Omega$ points outward.

Proof. We see that the boundary has four parts. We denote them by S_{xy}, S_{yz}, S_{zx} and S_4 where S_{xy} is the part of $\partial\Omega$ that is in the x-y plane, and so on. S_4 is the "top" surface in $x+y+z=1$.

We calculate the left hand side as follows.

- S_{xy}. As the outer normal is $(0,0,-1)$, we have

$$\begin{aligned}\int_{S_{xy}} f\,\mathrm{d}y \wedge \mathrm{d}z + g\,\mathrm{d}z \wedge \mathrm{d}x + h\,\mathrm{d}x \wedge \mathrm{d}y &= \int_{S_{xy}} \begin{pmatrix} f \\ g \\ h \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{d}S \\ &= -\int_D h(u,v,0)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}\tag{7}$$

  where $D := \{(u,v) |\, u \geqslant 0, v \geqslant 0, u+v \leqslant 1\}$.

- S_{yz}. Similarly we have

$$\int_{S_{yz}} f\,\mathrm{d}y \wedge \mathrm{d}z + g\,\mathrm{d}z \wedge \mathrm{d}x + h\,\mathrm{d}x \wedge \mathrm{d}y = -\int_D f(0,u,v)\,\mathrm{d}u\,\mathrm{d}v. \tag{8}$$

- S_{zx}. We have

$$\int_{S_{zx}} f\,\mathrm{d}y \wedge \mathrm{d}z + g\,\mathrm{d}z \wedge \mathrm{d}x + h\,\mathrm{d}x \wedge \mathrm{d}y = -\int_D g(u,0,v)\,\mathrm{d}u\,\mathrm{d}v. \tag{9}$$

- S_4. We parametrize S_4 as

$$\Phi(u,v) = (u, v, 1-u-v), \qquad (u,v) \in D. \tag{10}$$

  We have

$$\frac{\partial\Phi}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \frac{\partial\Phi}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Longrightarrow \frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{11}$$

As this vector points upward we see that it is an outer normal. Therefore

$$\begin{aligned}\int_{S_4} h\,\mathrm{d}x\wedge\mathrm{d}y &= \int_D \begin{pmatrix}0\\0\\h\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\mathrm{d}u\,\mathrm{d}v\\ &= \int_D h(u,v,1-u-v)\,\mathrm{d}u\,\mathrm{d}v. \qquad (12)\end{aligned}$$

Now we see that $\int_{S_4} f\,\mathrm{d}y\wedge\mathrm{d}z + \int_{S_{xy}} f\,\mathrm{d}y\wedge\mathrm{d}z + g\,\mathrm{d}z\wedge\mathrm{d}x + h\,\mathrm{d}x\wedge\mathrm{d}y$ equals

$$\begin{aligned}\int_D [h(u,v,1-u-v)\,\mathrm{d}u\,\mathrm{d}v - h(u,v,0)]\,\mathrm{d}u\,\mathrm{d}v &= \int_D\int_0^{1-u-v}\frac{\partial h}{\partial z}\,\mathrm{d}z\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_\Omega \frac{\partial h}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \qquad (13)\end{aligned}$$

Similarly we have (it's convenient to parametrize S_4 differently for each)

$$\int_{S_4} f\,\mathrm{d}y\wedge\mathrm{d}z + \int_{S_{yz}}\cdots = \int_\Omega \frac{\partial f}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (14)$$

and

$$\int_{S_4} g\,\mathrm{d}z\wedge\mathrm{d}x + \int_{S_{zx}}\cdots = \int_\Omega \frac{\partial g}{\partial y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (15)$$

and the conclusion follows. □