NAME: ID:

MATH 317 Q1 WINTER 2017 QUIZ 2 SOLUTIONS

Feb. 3, 2017, 25 minutes

- The quiz has 3 problems. Total 10 + 1 points.

QUESTION 1. (5 PTS) *Let* $y = Y(x)$ *be implicitly defined near the point* $(1, 2)$ *through the equation*

$$x^2 + 2\,x\,y - y^2 = 1. \tag{1}$$

Calculate $Y'(1)$ *and* $Y''(1)$.

Solution. We have

$$2\,x + 2\,Y + 2\,x\,Y' - 2\,YY' = 0. \tag{2}$$

Setting $x = 1, Y = 2$ we obtain

$$2 + 4 + 2\,Y' - 4\,Y' = 0 \Longrightarrow Y'(1) = 3. \tag{3}$$

Differentiating again, we have

$$2 + 4\,Y' + 2\,x\,Y'' - 2\,(Y')^2 - 2\,YY'' = 0 \tag{4}$$

which after setting $x = 1, Y = 2, Y' = 3$ gives

$$2 + 12 + 2\,Y'' - 18 - 4\,Y'' = 0 \Longrightarrow Y''(1) = -2. \tag{5}$$

QUESTION 2. (5 PTS) *Let $U(x,y), V(x,y)$ be defined implicitly through*

$$x\,u - y\,v = 0, \qquad y\,u + x\,v = 1. \tag{6}$$

Calculate its Jacobian matrix at $x = y = 1$.

Solution. We see that $f(x,y,u,v) = \begin{pmatrix} x\,u - y\,v \\ y\,u + x\,v - 1 \end{pmatrix}$. We calculate

$$\frac{\partial f}{\partial(x,y)} = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}, \qquad \frac{\partial f}{\partial(u,v)} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}. \tag{7}$$

We note that when $x = y = 1$, there holds $u - v = 0, u + v = 1$ so $U(1,1) = V(1, 1) = \frac{1}{2}$. Consequently at $(1,1)$, we have

$$J = -\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = -\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

QUESTION 3. (1 BONUS PT) *Use Lagrange multiplier theory to prove* $\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| \leqslant (a^2+b^2)^{1/2}\,(c^2+d^2)^{1/2}$.

Proof. We consider the following problem:

$$\max\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{subject to } a^2+b^2=c^2+d^2=1. \tag{9}$$

Note that as $\det(-A)=-\det A$, $\max\det A=\max|\det A|$. We form the Lagrange function

$$\mathcal{L}(a,b,c,d,\lambda,\mu)=\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\lambda\,(a^2+b^2-1)+\mu\,(c^2+d^2-1). \tag{10}$$

Taking partial derivatives:

$$\begin{aligned} 0=\frac{\partial\mathcal{L}}{\partial a} &= d+2\,a\,\lambda; \\ 0=\frac{\partial\mathcal{L}}{\partial b} &= -c+2\,b\,\lambda; \\ 0=\frac{\partial\mathcal{L}}{\partial c} &= -b+2\,c\,\mu; \\ 0=\frac{\partial\mathcal{L}}{\partial d} &= a+2\,d\,\mu. \end{aligned}$$

We solve that $c^2+d^2=4\,\lambda^2\,(a^2+b^2) \Longrightarrow \lambda^2=\frac{1}{4}$. Similarly $\mu^2=\frac{1}{4}$. Thus we see that $|d|=|a|, |c|=|b|$. Consequently at a stationary point,

$$\det A=a\,d-b\,c\leqslant a^2+b^2=1. \tag{11}$$

In particular this holds at the maximum. Thus ends the proof. $\square$