Name: ID:

# Math 317 Q1 Winter 2017 Quiz 1 Solutions

Jan. 20, 2017, 25 minutes

- The quiz has 3 problems. Total 10 + 1 points.

Question 1. (5 pts) *Let* $S := \left\{ \sin\left(\frac{1}{n}\right) | \, n \in \mathbb{N} \right\}$. *Find all cluster points of* $S$. *Justify your claim.*

**Solution.** We claim that the only cluster point is 0.

- 0 is a cluster point of $S$.

  Let $\varepsilon > 0$ be arbitrary. Take $n \in \mathbb{N}$ be such that $\frac{1}{n} < \varepsilon$. Then we have

$$\sin\left(\frac{1}{n}\right) \in S, \qquad 0 < \left| \sin\left(\frac{1}{n}\right) - 0 \right| \leqslant \left| \frac{1}{n} - 0 \right| < \varepsilon. \tag{1}$$

- Let $a \neq 0$. Then $a$ is not a cluster point of $S$.
  - If $a < 0$, set $\varepsilon_0 = |a|$. Then we have $S \cap B(a, \varepsilon_0) = \varnothing$, thus $a$ is not a cluster point of $S$.
  - If $a > \sin 1$, set $\varepsilon_0 = |a - \sin 1| > 0$. Then we have $S \cap B(a, \varepsilon_0) = \varnothing$ and $a$ is not a cluster point of $S$.
  - If $a = \sin 1$, set $\varepsilon_0 = |a - \sin(1/2)| > 0$. Then we have $S \cap B(a, \varepsilon_0) = \varnothing$ and $a$ is not a cluster point of $S$.
  - If $0 < a < \sin 1$, there is $n_0 \in \mathbb{N}$ such that $\sin\left(\frac{1}{n_0}\right) < a$ while $\sin\left(\frac{1}{n_0 - 1}\right) > a$. Set $\varepsilon_0 = \min \{|a - \sin(1/n_0)|, |a - \sin(1/(n_0 - 1))|\}$. Then we have $S \cap B(a, \varepsilon_0) = \varnothing$ and $a$ is not a cluster point of $S$.

QUESTION 2. (5 PTS) *Calculate all first and second order partial derivatives of* $f(x,y) := e^{xy}\cos(y)$.

**Solution.**

We have

$$\frac{\partial f}{\partial x} = y\,e^{xy}\cos y; \qquad \frac{\partial f}{\partial y} = -e^{xy}\sin y + x\,e^{xy}\cos y = e^{xy}\,(x\cos y - \sin y); \tag{2}$$

$$\frac{\partial^2 f}{\partial x^2} = y^2\,e^{xy}\cos y; \tag{3}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = e^{xy}\,(\cos y + x\,y\cos y - y\sin y); \tag{4}$$

$$\frac{\partial^2 f}{\partial y^2} = e^{xy}\,(x^2\cos y - 2\,x\sin y - \cos y). \tag{5}$$

QUESTION 3. (1 BONUS PT) *Let* $A := \left\{(x, y) | \, 0 \leqslant x \leqslant 1, \, y = \frac{1}{q}$ *if* $x \in \mathbb{Q}, x = \frac{p}{q}$ *in lowest terms, and* $y = 0$ *if* $x \notin \mathbb{Q}. \right\}$ *Prove that* $A$ *has zero content.*

**Proof.** Let $\varepsilon > 0$ be arbitrary. Let $I_0 := [0, 1] \times [0, \varepsilon / 2]$. Then we have

$$A \backslash I_0 = \left\{(x, y) | \, x \in \mathbb{Q} \cap [0, 1], x = \frac{p}{q}, q < \frac{2}{\varepsilon}, y = \frac{1}{q}\right\}. \tag{6}$$

Since $0 \leqslant p < q$, we see that $A \backslash I_0$ is a finite set. Denote $A \backslash I_0 = \{(x_i, y_i) | \, i = 1, 2, \ldots, N\}$. We see that

$$A \backslash I_0 \subseteq \cup_{i=1}^N I_i \tag{7}$$

where $I_i := (x_i, y_i)$. Thus we see that

$$A \subseteq \cup_{i=0}^N I_i \tag{8}$$

and $\sum_{i=0}^N |I_i| = \frac{\varepsilon}{2} < \varepsilon$. As $\varepsilon$ is arbitrary, this means $A$ has zero content. $\square$