We evaluate the integral

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x. \tag{1}$$

Consider the function

$$g(y) := \int_0^\infty e^{-xy}\,\frac{\sin x}{x}\,\mathrm{d}x. \tag{2}$$

We claim that:

- $g(y)$ is defined for all $y > 0$.

  Let $y > 0$ be arbitrary. We first prove $\left|\frac{\sin x}{x}\right| \leqslant 1$ for all $x \in [0, \infty)$. We apply MVT:

  $$\left|\frac{\sin x}{x}\right| = \left|\frac{\sin x - \sin 0}{x - 0}\right| = |\cos c| \leqslant 1. \tag{3}$$

  Thus we have

  $$\left|e^{-xy}\,\frac{\sin x}{x}\right| \leqslant e^{-xy}. \tag{4}$$

  Next we prove e^{-xy} is improperly integrable on $(0, \infty)$. As e^{-xy} is integrable on $[0, d]$ for all $d > 0$, we calculate

  $$\int_0^d e^{-xy}\,\mathrm{d}x = \frac{1}{y}\int_0^d e^{-xy}\,\mathrm{d}(x\,y) = \frac{1 - e^{-dy}}{y} \longrightarrow \frac{1}{y} \text{ as } d \to \infty. \tag{5}$$

  The conclusion now follows.

- $\lim_{y\to\infty} g(y) = 0$.

  From the previous calculation we have

  $$|g(y)| \leqslant \int_0^\infty e^{-xy}\,\mathrm{d}x = \frac{1}{y}. \tag{6}$$

  Thus $\lim_{y\to\infty} g(y) = 0$ follows from Squeeze Theorem.

- $g(y)$ is differentiable on $(0, \infty)$ with $g'(y) = -\frac{1}{1+y^2}$.

  Let $y_0 \in (0, \infty)$ be arbitrary. Denote $A := -\int_0^\infty e^{-xy_0}\sin x\,\mathrm{d}x = -\frac{1}{1+y_0^2}$.

  - Let $h \in (0, y_0)$. We have

    $$\frac{g(y_0 + h) - g(y_0)}{h} - A = \int_0^\infty \frac{e^{-xh} - 1 + x\,h}{x\,h}\,e^{-xy_0}\sin x\,\mathrm{d}x. \tag{7}$$

    By MVT we have

    $$\left|\frac{e^{-xh} - 1}{x\,h}\right| = \left|\frac{e^{-xh} - e^{-0}}{x\,h - 0}\right| = |e^{-c}| \tag{8}$$

    for some $c \in (0, x\,h)$ which means

    $$\left|\frac{e^{-xh} - 1 + x\,h}{x\,h}\right| \leqslant 2 \tag{9}$$

    for all $x > 0$.

    Next by Taylor expansion with Lagrange form of remainder we have

    $$e^{-xh} = 1 - x\,h + \frac{e^{-c}}{2!}\,x^2\,h^2 \tag{10}$$

    for some $c \in (0, x\,h)$. Therefore

    $$\left|\frac{e^{-xh} - 1 + x\,h}{x\,h}\right| \leqslant x\,h \tag{11}$$

for all $x > 0$. Now we estiamte

$$\begin{aligned}\left|\frac{g(y_0+h)-g(y_0)}{h}-A\right| &\leqslant \left|\int_0^{h^{-1/2}} \frac{e^{-xh}-1+x\,h}{x\,h}\,e^{-xy_0}\sin x\,\mathrm{d}x\right| \\ &\quad+\left|\int_{h^{-1/2}}^{\infty} \frac{e^{-xh}-1+x\,h}{x\,h}\,e^{-xy_0}\sin x\,\mathrm{d}x\right| \\ &\leqslant h\int_0^{h^{-1/2}} x\,e^{-xy_0}\,\mathrm{d}x \\ &\quad+\int_{h^{-1/2}}^{\infty} 2\,e^{-xy_0}\,\mathrm{d}x. \end{aligned} \tag{12}$$

It is clear that

$$\lim_{h\to 0+}\int_{h^{-1/2}}^{\infty} 2\,e^{-xy_0}\,\mathrm{d}x = 0. \tag{13}$$

On the other hand, we have

$$\int_0^{h^{-1/2}} x\,e^{-xy_0}\,\mathrm{d}x = y_0^{-2}\int_0^{h^{-1/2}y_0} u\,e^{-u}\,\mathrm{d}u \leqslant y_0^{-2}\left(h^{-1/2}\,y_0+1\right). \tag{14}$$

Therefore

$$\lim_{h\to 0+} h\int_0^{h^{-1/2}} x\,e^{-xy_0}\,\mathrm{d}x = 0. \tag{15}$$

Consequently

$$\lim_{h\to 0+}\frac{g(y_0+h)-g(y_0)}{h} = A. \tag{16}$$

- Let $h \in \left(-\frac{y_0}{2}, 0\right)$. We have

$$\frac{g(y_0+h)-g(y_0)}{h} - A = \int_0^{\infty} \frac{1+x\,h\,e^{xh}-e^{xh}}{x\,h}\,e^{-x(y_0+h)}\sin x\,\mathrm{d}x. \tag{17}$$

Similar to the $h > 0$ case, we have

$$\left|\frac{1+x\,h\,e^{xh}-e^{xh}}{x\,h}\right| \leqslant 2, \qquad \left|\frac{1+x\,h\,e^{xh}-e^{xh}}{x\,h}\right| \leqslant x\,|h|. \tag{18}$$

Furthermore we have

$$e^{-x(y_0+h)} \leqslant e^{-xy_0/2}. \tag{19}$$

Now the proof proceeds similar to the case $h > 0$ and we conclude

$$\lim_{h\to 0-}\frac{g(y_0+h)-g(y_0)}{h} = A. \tag{20}$$

- Finally,

$$\lim_{y\to 0+} g(y) = \int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x. \tag{21}$$

Denote $I := \int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x$. We consider

$$g(y) - I = \int_0^{\infty} (e^{-xy}-1)\,\frac{\sin x}{x}\,\mathrm{d}x. \tag{22}$$

Define $h(x) := \frac{e^{-xy}-1}{x}$. Then

$$g(y) - I = \int_0^{\infty} h(x)\sin x\,\mathrm{d}x. \tag{23}$$

First notice that $\lim_{x\to\infty} h(x) = 0$ as $x \longrightarrow \infty$. Furthermore we calculate

$$h'(x) = \frac{-x\,y\,e^{-xy} + 1 - e^{-xy}}{x^2} = \frac{e^{-xy}}{x^2}\,[e^{xy} - 1 - x\,y] > 0 \text{ for all } x, y > 0. \tag{24}$$

Thus $h(x)$ increases from $-y$ to 0 as x runs from 0 to ∞.

We calculate

$$\begin{aligned} |g(y) - I| = \left|\int_0^\infty h(x)\sin x\,\mathrm{d}x\right| &= \left|-h(x)\cos x|_0^\infty + \int_0^\infty h'(x)\cos x\,\mathrm{d}x\right| \\ &\leqslant |h(0+)| + \int_0^\infty |h'(x)|\,\mathrm{d}x \\ &= 2\,|h(0+)| = 2\,y. \end{aligned} \tag{25}$$

Now it is obvious that $\lim_{y\to 0+} [g(y) - I] = 0$.

$g'(y) = -\frac{1}{1+y^2}$ together with $\lim_{y\to\infty} g(y) = 0$ gives $g(y) = \frac{\pi}{2} - \arctan y$. Therefore we have

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = g(0+) = \frac{\pi}{2}, \tag{26}$$