Example 1. Calculate

$$\int_L |x\,y|\,\mathrm{d}s \tag{1}$$

with L: $\begin{pmatrix} \cos^3 t \\ \sin^3 t \end{pmatrix}$, $t \in [0, 2\,\pi]$.

Solution. We have

$$\begin{aligned}
\int_L |x\,y|\,\mathrm{d}s &= \int_0^{2\pi} |(\cos^3 t)\,(\sin^3 t)|\,\sqrt{(-3\cos^2 t\sin t)^2 + (3\sin^2 t\cos t)^2}\,\mathrm{d}t \\
&= \int_0^{2\pi} |\cos t\sin t|^3 \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}\,\mathrm{d}t \\
&= \int_0^{2\pi} |\cos t\sin t|^3\,(3)\,|\cos t\sin t| \\
&= 3\int_0^{2\pi} (\cos t\sin t)^4\,\mathrm{d}t \\
&= \frac{3}{16}\int_0^{2\pi} (\sin 2\,t)^4\,\mathrm{d}t \\
&= \frac{3}{16}\int_0^{2\pi} \left(\frac{1-\cos 4\,t}{2}\right)^2 \mathrm{d}t \\
&= \frac{3\,\pi}{32} + \frac{3}{64}\int_0^{2\pi} (\cos 4\,t)^2\,\mathrm{d}t \\
&= \frac{9\,\pi}{64}.
\end{aligned} \tag{2}$$

Example 2. Calculate

$$\int_S (x + y + z)\,\mathrm{d}S \tag{3}$$

with S the first octant part of the unit sphere: $x^2 + y^2 + z^2 = 1$, $x, y, z \geqslant 0$.

Solution. We parametrize S:

$$\begin{pmatrix} \cos\varphi\cos\psi \\ \sin\varphi\cos\psi \\ a\sin\psi \end{pmatrix}, \qquad \varphi \in \left[0, \frac{\pi}{2}\right], \qquad \psi \in \left[0, \frac{\pi}{2}\right]. \tag{4}$$

Now calculate

$$\frac{\partial\Phi}{\partial\varphi} = \begin{pmatrix} -\sin\varphi\cos\psi \\ \cos\varphi\cos\psi \\ 0 \end{pmatrix}, \qquad \frac{\partial\Phi}{\partial\psi} = \begin{pmatrix} -\cos\varphi\sin\psi \\ -\sin\varphi\sin\psi \\ \cos\psi \end{pmatrix}, \tag{5}$$

we have

$$\left\|\frac{\partial\Phi}{\partial\varphi} \times \frac{\partial\Phi}{\partial\psi}\right\| = |\cos\psi| = \cos\psi. \tag{6}$$

Thus we have

$$\begin{aligned}
\int_S (x+y+z)\,\mathrm{d}S &= \int_{[0,\frac{\pi}{2}]^2} [\cos\varphi\cos\psi + \sin\varphi\cos\psi + \sin\psi]\,a^2\cos\psi\,\mathrm{d}(\varphi,\psi) \\
&= \int_0^{\pi/2} \left[\int_0^{\pi/2} (\cos\varphi + \sin\varphi)\,(\cos\psi)^2 + \cos\psi\sin\psi\,\mathrm{d}\psi\right] \mathrm{d}\varphi \\
&= \int_0^{\pi/2} \frac{\pi}{4}\,(\cos\varphi + \sin\varphi) + \frac{1}{2}\,\mathrm{d}\varphi \\
&= \frac{3\,\pi}{4}.
\end{aligned} \tag{7}$$

Example 3. Calculate

$$\int_L y\,\mathrm{d}x - z\,\mathrm{d}y + x\,\mathrm{d}z \tag{8}$$

where L is the intersection of $(x^2+y^2)/2+z^2=1$ and $x=y$, oriented counter-clockwise when viewed from the positive x-axis.

Solution. We notice that on L, $x=y=t$, $t^2+z^2=1$. Thus we parametrize L as

$$(\cos\theta, \cos\theta, \sin\theta), \qquad \theta\in[0,2\pi]. \tag{9}$$

Note that θ is the angle from the x-y plane to the point on L, therefore the orientation $0\longrightarrow 2\pi$ is consistent with the specified orientation.

Thus we have

$$\begin{aligned}\int_L y\,\mathrm{d}x - z\,\mathrm{d}y + x\,\mathrm{d}z &= \int_0^{2\pi} [\cos\theta\,(a\cos\theta)' - \sin\theta)\,(\cos\theta)' + (\cos\theta)\,(\sin\theta)']\,\mathrm{d}\theta \\ &= \int_0^{2\pi} [-\cos\theta\sin\theta + (\sin\theta)^2 + (\cos\theta)^2]\,\mathrm{d}\theta \\ &= 2\pi. \end{aligned} \tag{10}$$

Example 4. Calculate

$$\int_S \begin{pmatrix} x^2 \\ -y^2 \\ z^2 \end{pmatrix} \cdot \mathrm{d}S \tag{11}$$

where $S=\partial V$ where $V=\{x^2+y^2+z^2\leqslant 3\}\cap\{z\geqslant 0\}\cap\left\{z\geqslant\sqrt{x^2+y^2-1}\right\}$, oriented by the outer normal.

Solution. $S=S_{\text{bottom}}+S_{\text{side}}+S_{\text{top}}$ with

- Bottom: $x^2+y^2\leqslant 1$, $z=0$;
- Side: $1\leqslant x^2+y^2\leqslant 2$, $z=\sqrt{x^2+y^2-1}$;
- Top: $x^2+y^2\leqslant 2$, $z=\sqrt{3-x^2-y^2}$.

We calculate the integral on each one by one.

- Bottom.

  The natural parametrization is $\begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$; the outer normal is $\boldsymbol{n}_{\text{bottom}}=\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. Thus

$$\int_{S_{\text{bottom}}} = \int_{u^2+v^2\leqslant 1} \begin{pmatrix} u^2 \\ -v^2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{d}S = 0. \tag{12}$$

- Side. We parametrize the side as $\begin{pmatrix} u \\ v \\ \sqrt{u^2+v^2-1} \end{pmatrix}$ with $D=\{1\leqslant u^2+v^2\leqslant 2\}$. Then calculate

$$\boldsymbol{r}_u = \begin{pmatrix} 1 \\ 0 \\ \frac{u}{\sqrt{u^2+v^2-1}} \end{pmatrix}, \qquad \boldsymbol{r}_v = \begin{pmatrix} 0 \\ 1 \\ \frac{v}{\sqrt{u^2+v^2-1}} \end{pmatrix} \Longrightarrow \boldsymbol{r}_u\times\boldsymbol{r}_v = \begin{pmatrix} -\frac{u}{\sqrt{u^2+v^2-1}} \\ -\frac{v}{\sqrt{u^2+v^2-1}} \\ 1 \end{pmatrix}. \tag{13}$$

Note that the outer normal should point down therefore we should use $-\boldsymbol{r}_u \times \boldsymbol{r}_v$ and have

$$\int_{S_{\text{side}}} = \int_D \begin{pmatrix} u^2 \\ -v^2 \\ u^2+v^2-1 \end{pmatrix} \cdot \begin{pmatrix} \frac{u}{\sqrt{u^2+v^2-1}} \\ \frac{v}{\sqrt{u^2+v^2-1}} \\ -1 \end{pmatrix} \mathrm{d}(u,v). \tag{14}$$

Noticing D is symmetric in u, v we see that

$$\begin{aligned} \int_{S_{\text{side}}} &= \int_D \frac{u^3}{\sqrt{u^2+v^2-1}} \mathrm{d}(u,v) \\ &\quad -\int_D \frac{v^2}{\sqrt{u^2+v^2-1}} \mathrm{d}(u,v) \\ &\quad +\int_D (1-u^2-v^2) \mathrm{d}(u,v) \\ &= 0-0+2\pi \int_1^{\sqrt{2}} (1-r^2)\, r\, \mathrm{d}r \\ &= -\frac{\pi}{2}; \end{aligned} \tag{15}$$

- Top. We parametrize $\begin{pmatrix} u \\ v \\ \sqrt{3-u^2-v^2} \end{pmatrix}$ with $D = \{u^2+v^2 \leqslant 2\}$. Then calculate

$$\boldsymbol{r}_u = \begin{pmatrix} 1 \\ 0 \\ -\frac{u}{\sqrt{3-u^2-v^2}} \end{pmatrix}, \quad \boldsymbol{r}_v = \begin{pmatrix} 0 \\ 1 \\ -\frac{v}{\sqrt{3-u^2-v^2}} \end{pmatrix} \Longrightarrow \boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{pmatrix} \frac{u}{\sqrt{3-u^2-v^2}} \\ \frac{v}{\sqrt{3-u^2-v^2}} \\ 1 \end{pmatrix}. \tag{16}$$

This time it is consistent with the outer normal. Again noticing the symmetry, we have

$$\begin{aligned} \int_{S_{\text{top}}} &= \int_D (3-u^2-v^2)\, \mathrm{d}(u,v) \\ &= 2\pi \int_0^{\sqrt{2}} (3-r^2)\, r\, \mathrm{d}r \\ &= 4\pi. \end{aligned} \tag{17}$$

Adding things up we have $7\pi/2$.