Arc length and surface area

Example 1. Calculate the arc length of the space curve

$$x = \cos t, y = \sin t, z = t \tag{1}$$

for t from 0 to 2π.
Solution. We have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, \mathrm{d}t \\ &= \int_0^{2\pi} \sqrt{2}\, \mathrm{d}t = 2\sqrt{2}\,\pi. \end{aligned} \tag{2}$$

Example 2. (Cycloid) The trajectory of a unit circle rolling in the plane along a line.

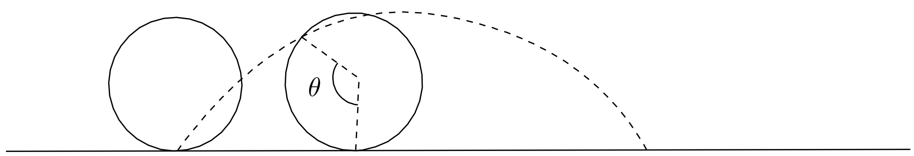

Figure 1. The cycloids

We take $t = \theta$ as the parameter. We have $x(t) = (t - \sin t, 1 - \cos t)$ for $t \in [0, 2\pi]$. For its arc length, we have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, \mathrm{d}t \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, \mathrm{d}t \\ &= \int_0^{2\pi} \sqrt{2 - 2\left(1 - 2\left(\sin\frac{t}{2}\right)^2\right)}\, \mathrm{d}t \\ &= 2\int_0^{2\pi} \left|\sin\left(\frac{t}{2}\right)\right| \mathrm{d}t \\ &= 8. \end{aligned} \tag{3}$$

Example 3. Find the area of the part of $z = x\,y$ that is inside $x^2 + y^2 = 1$.

Solution. We calculate

$$S = \int_{x^2+y^2 \leqslant 1} \sqrt{1 + z_x^2 + z_y^2}\, \mathrm{d}(x, y) = \frac{2\pi}{3}\left(2\sqrt{2} - 1\right). \tag{4}$$

Example 4. Find the surface area of the sphere $x^2 + y^2 + z^2 = R^2$.

Solution. We use the parametrization

$$\sigma(\phi, \psi) = \begin{pmatrix} R\cos\phi\cos\psi \\ R\sin\phi\cos\psi \\ R\sin\psi \end{pmatrix}, \qquad U = \left\{(\phi, \psi) \,|\, 0 < \phi < 2\pi, -\frac{\pi}{2} < \psi < \frac{\pi}{2}\right\}. \tag{5}$$

Then calculate

$$\sigma_u = \begin{pmatrix} -R\sin\phi\cos\psi \\ R\cos\phi\cos\psi \\ 0 \end{pmatrix}, \qquad \sigma_v = \begin{pmatrix} -R\cos\phi\sin\psi \\ -R\sin\phi\sin\psi \\ R\cos\psi \end{pmatrix}, \tag{6}$$

$$\|\sigma_u \times \sigma_v\| = R^2\cos\psi. \tag{7}$$

This gives

$$S = \int_U R^2 \cos\psi \, \mathrm{d}(\phi, \psi) = 4\,\pi\,R^2. \tag{8}$$

Line and surface integrals of the first kind

Line integral of the first kind

Let γ be the trace of a regular curve $x(t){:}\,[a,b] \mapsto \mathbb{R}^N$. Let $f{:}\,\gamma \mapsto \mathbb{R}$ be a continuous function. Then we can define the line integral of the first kind as

$$\int_\gamma f \, \mathrm{d}s := \int_a^b f(x(t))\, \|x'(t)\| \, \mathrm{d}t. \tag{9}$$

Surface integral of the first kind

Let S be a C^1 surface parametrized through $\Phi(u,v){:}\,U \mapsto \mathbb{R}^3$. Let $f{:}\,S \mapsto \mathbb{R}$ be a continuous function. Then we can define the surface integral of the first kind as

$$\int_S f \, \mathrm{d}S := \int_U f(\Phi(u,v)) \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| \mathrm{d}u \, \mathrm{d}v. \tag{10}$$

Examples

Example 5. Calculate $\int_\gamma x^2 \, \mathrm{d}s$ where γ is the unit circle.

Solution. We parametrize γ as $(\cos t, \sin t)$ for $t \in [0, 2\,\pi]$. Then

$$\int_\gamma x^2 \, \mathrm{d}s = \int_0^{2\pi} (\cos t)^2 \, \mathrm{d}t = \pi. \tag{11}$$

Example 6. Calculate

$$\int_S z^2 \, \mathrm{d}S \tag{12}$$

where $S := \{(x, y, z) |\, x^2 + y^2 + z^2 = R^2\}$.

Solution. We parametrize S:

$$\boldsymbol{r}(\phi, \psi) = \begin{pmatrix} R\cos\phi\cos\psi \\ R\sin\phi\cos\psi \\ R\sin\psi \end{pmatrix}, \qquad D = \left\{ (\phi, \psi) |\, 0 \leqslant \phi \leqslant 2\,\pi, -\frac{\pi}{2} \leqslant \psi \leqslant \frac{\pi}{2} \right\}. \tag{13}$$

Then we have

$$r_\phi = R \begin{pmatrix} -\sin\phi\cos\psi \\ \cos\phi\cos\psi \\ 0 \end{pmatrix}, \qquad r_\psi = R \begin{pmatrix} -\cos\phi\sin\psi \\ -\sin\phi\sin\psi \\ \cos\psi \end{pmatrix} \tag{14}$$

and

$$E = R^2\cos^2\psi, \quad F = 0, \quad G = R^2 \Longrightarrow \sqrt{E\,F - G^2} = R^2\,|\cos\psi| = R^2\cos\psi. \tag{15}$$

Note that the last equality follows from the fact that $-\frac{\pi}{2} \leqslant \psi \leqslant \frac{\pi}{2}$.

Now we calculate

$$\begin{aligned} I &= \int_D (R\sin\psi)^2\,R^2\cos\psi \\ &= R^4 \int_D (\sin\psi)^2 \cos\psi \, \mathrm{d}\psi \\ &= R^4 \int_0^{2\pi} \left[\int_{-\pi/2}^{\pi/2} (\sin\psi)^2 \cos\psi \, \mathrm{d}\psi \right] \mathrm{d}\phi \\ &= R^4 \int_0^{2\pi} \left[\frac{(\sin\psi)^3}{3} \right]_{-\pi/2}^{\pi/2} \mathrm{d}\phi \\ &= R^4 \int_0^{2\pi} \frac{2}{3}\, \mathrm{d}\phi = \frac{4\,\pi\,R^4}{3}. \end{aligned} \tag{16}$$

Remark 7. The calculation can be much simplified through the following trick:

By symmetry we see that

$$\int_S z^2 \,\mathrm{d}S = \frac{1}{3} \int_S (x^2 + y^2 + z^2) \,\mathrm{d}S = \frac{1}{3} \int_S \mathrm{d}S = \frac{4\,\pi}{3}. \tag{17}$$

Remark 8. For more examples, please see my lecture notes for 2014 Math 317.