Hadamard's global inverse function theorem

Theorem 1. (Hadamard) *Let $F:\mathbb{R}^N \mapsto \mathbb{R}^N$ be a C^2 mapping. Suppose that $F(0)=0$ and that the Jacobian determinant of F is nonzero at each point. Further suppose that F is proper. Then F is one-to-one and onto.*

Remark 2. (Proper mappings) F is proper if whenever K is compact, so is $F^{-1}(K)$.

> **Exercise 1.** show that there are discontinuous functions that are proper, and also continuous functions that are not proper.
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> **Exercise 2.** Let $F:\mathbb{R}^N \mapsto \mathbb{R}^N$ be continuous. Assume that whenever B is bounded, so is $F^{-1}(B)$. Prove or disprove: F is proper.

Proof. Define $H:\mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}^N$ as

$$H(t,x)=\begin{cases} F(t\,x)/t & t>0 \\ DF(0)(x)+\frac{1}{2}\,D^2F(0)(x,x)\,t & t\leqslant 0 \end{cases}. \tag{1}$$

We notice the following.

$$H(0,x)=D\,F(0)(x), \qquad H(1,x)=F(x). \tag{2}$$

Furthermore we have $H\in C^1$. For $t>0$ we calculate

$$\frac{\partial H}{\partial x_i}(t,x)=\frac{\partial F}{\partial x_i}(t\,x), \qquad \frac{\partial H}{\partial t}(t,x)=\frac{D\,F(t\,x)(x)}{t}-\frac{F(t\,x)}{t^2}. \tag{3}$$

Applying Taylor's theorem, we have (recall that $F(0)=0$)

$$\begin{aligned} \frac{\partial H}{\partial t}(t,x) &= \frac{D\,F(t\,x)(x)}{t}-\frac{F(t\,x)}{t^2} \\ &= \frac{1}{t^2}\left[D\,F(t\,x)\,(t\,x)-F(t\,x)\right] \\ &= \frac{1}{t^2}\left[(D\,F(0)+D^2F(0)(t\,x)+o(t))(t\,x)-\left(D\,F(0)(t\,x)+\frac{1}{2}\,D^2F(0)(t\,x,t\,x)+o(t^2)\right)\right] \\ &= \frac{1}{2}\,D^2F(0)(x,x)+o(1). \end{aligned} \tag{4}$$

Thus $H\in C^1$.

By assumption $H(0,x)$ is a bijection on $\mathbb{R}^N$. Let $y\in\mathbb{R}^N$ be arbitrary. We consider the set $C(y):=\{(t,x)|\ H(t,x)=y, t\geqslant 0\}$. As $H(0,x)$ is a bijection, $C(y)$ is not empty. Intuitively $C(y)$ is a curve. All we need to show now is that this curve intersects the hyperplane $t=1$. In the following we turn this idea into a rigorous argument.

Fix $\bar{y}\in\mathbb{R}^N$. Define

$$f(t,x):=H(t,x)-\bar{y}. \tag{5}$$

Let $\bar{x}$ be such that $D\,F(0)(\bar{x})=\bar{y}$. Let $C:=\{(t,x)|\ f(t,x)=0\}$.

I. $C(y)$ is a union of closed C^1 curves.

Since $(0,\bar{x})\in C$, C is nonempty. Now let $(t_0,x_0)\in C$ be arbitrary. We have

$$\frac{\partial f}{\partial x}(t_0,x_0)=J_F(t_0\,x_0) \tag{6}$$

which is non-singular. By the implicit function theorem there is a $\varepsilon_0>0$ such that $C\cap B_\varepsilon((t_0,x_0))$ is of the form $x=\phi(t)$.

II. Each curve in $C(y)$ intersects $t=1$. Consequently $F:\mathbb{R}^N\mapsto\mathbb{R}^N$ is onto.

Assume not. Then there is a plane $t=t_0$ such that $\phi(t)\longrightarrow\infty$ as $t\searrow t_0$ or $t\nearrow t_0$. But $f(t,\phi(t))=\bar{y}$ which gives

$$F(t\,\phi(t))=\bar{y}\,t \tag{7}$$

as $t\longrightarrow t_0$, while $|\phi(t)|\longrightarrow\infty$. But this implies $F^{-1}(\overline{B_r(\bar{y}\,t_0)})$ is unbounded for some finite r. This contradicts the asumption that F is proper.

III. $F\colon \mathbb{R}^N \mapsto \mathbb{R}^N$ is one-to-one.

Assume the contrary. Let $x_1, x_2 \in \mathbb{R}^N$ be such that $F(x_1) = F(x_2) = \bar{y}$, that is $H(1, x_1) = H(1, x_2) = \bar{y}$. Consider the two curves emanating from $(1, x_1)$ and $(1, x_2)$ whose existences are guaranteed by the implicit function theorem. Similar argument as above shows that the two curves much intersect $t = 0$. As $DF(0)$ is nonsingular, the two curves must meet at the same point x_0 at $t = 0$. But this leads to a “pitchfork” bifurcation which is again prohibited by the implicit function theorem, applied at $(0, x_0)$ for the function $f(t, x)$.

Thus ends the proof. □