Example 1. Calculate y' and y'' at $x=0$ for the function implicitly defined through

$$x\,y-\ln y=0. \tag{1}$$

Solution. First we see that, when $x=0$, we have $\ln y=0$ so $y=1$. Thus all the calculation must be done at $(0,1)$. Taking derivatives we have

$$y+x\,y'-\frac{y'}{y}=0\xrightarrow{x=0,\,y=1}y'(0)=1. \tag{2}$$

Taking derivative again we have

$$y'+y'+x\,y''-\frac{y''}{y}+\frac{(y')^2}{y^2}=0\xrightarrow{x=0,\,y=1,\,y'=1}y''(0)=3. \tag{3}$$

Example 2. Calculate the formula of the Jacobian of (u,v) as implicitly defined function of (x,y) through

$$u^2-v=3\,x+y; \qquad u-2\,v^2=x-2\,y. \tag{4}$$

Solution. We have

$$f(x,y,u,v)=\left(\begin{array}{c} u^2-v-3\,x-y \\ u-2\,v^2-x+2\,y \end{array}\right). \tag{5}$$

We calculate

$$\frac{\partial f}{\partial(x,y)}=\left(\begin{array}{cc} -3 & -1 \\ -1 & 2 \end{array}\right), \qquad \frac{\partial f}{\partial(u,v)}=\left(\begin{array}{cc} 2\,u & -1 \\ 1 & -4\,v \end{array}\right). \tag{6}$$

Therefore

$$\begin{aligned}\frac{\partial(u,v)}{\partial(x,y)} &= -\left(\begin{array}{cc} 2\,u & -1 \\ 1 & -4\,v \end{array}\right)^{-1}\left(\begin{array}{cc} -3 & -1 \\ -1 & 2 \end{array}\right) \\ &= \frac{1}{1-8\,u\,v}\left(\begin{array}{cc} -4\,v & 1 \\ -1 & 2\,u \end{array}\right)\left(\begin{array}{cc} -3 & -1 \\ -1 & 2 \end{array}\right) \\ &= \frac{1}{1-8\,u\,v}\left(\begin{array}{cc} 12\,v-1 & 4\,v+2 \\ 3-2\,u & 1+4\,u \end{array}\right). \end{aligned} \tag{7}$$

Remark 3. Notice that $f(0,0,u,v)=\left(\begin{smallmatrix} u^2-v \\ u-2\,v^2 \end{smallmatrix}\right)$. Setting this to $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ we have two solutions $u=v=0$ and $u=2^{-1/3}$ and $v=2^{-2/3}$. Thus if we are talking about the implicit function defined near $x=y=u=v=0$, we have

$$J(0,0)=\left(\begin{array}{cc} -1 & 2 \\ 3 & 1 \end{array}\right); \tag{8}$$

But if we are talking about the implicit function defined near $x=y=0, u=2^{-1/3}, v=2^{-2/3}$, then

$$J(0,0)=-\frac{1}{3}\left(\begin{array}{cc} 12\cdot 2^{-2/3}-1 & 4\cdot 2^{-2/3}+2 \\ 3-2\cdot 2^{-1/3} & 1+4\cdot 2^{-1/3} \end{array}\right). \tag{9}$$

Example 4. Let $l,m,n\in\mathbb{N}$ and $a,b,c,k\in\mathbb{R}$. Further assume that $a,b,c\neq 0$. Solve

$$\max x^l\,y^m\,z^n \qquad \text{subject to } x^2+y^2+z^2=1. \tag{10}$$

Solution. We form the Lagrange function

$$\mathcal{L}(x,y,z,\lambda):=x^l\,y^m\,z^n+\lambda\,(x^2+y^2+z^2-1). \tag{11}$$

Taking partial derivatives we have

$$\frac{\partial\mathcal{L}}{\partial x} = l\,x^{l-1}\,y^m\,z^n+2\,\lambda\,x; \tag{12}$$

$$\frac{\partial\mathcal{L}}{\partial y} = m\,x^l\,y^{m-1}\,z^n+2\,\lambda\,y; \tag{13}$$

$$\frac{\partial\mathcal{L}}{\partial z} = n\,x^l\,y^m\,z^{n-1}+2\,\lambda\,z. \tag{14}$$

Setting them to zero we have

$$-2\,\lambda = l\,x^{l-2}\,y^m\,z^n = m\,x^l\,y^{m-2}\,z^n = n\,x^l\,y^m\,z^{n-2}. \tag{15}$$

Dividing (15) by $x^l\,y^m\,z^n$ we see that

$$\frac{l}{x^2} = \frac{m}{y^2} = \frac{n}{z^2}. \tag{16}$$

Together with $x^2 + y^2 + z^2 = 1$ we have

$$x = \left(\frac{l}{l+m+n}\right)^{1/2}, \quad y = \left(\frac{m}{l+m+n}\right)^{1/2}, \quad z = \left(\frac{n}{l+m+n}\right)^{1/2} \tag{17}$$

and the maximum is given by

$$\left[\frac{l^l\,m^m\,n^n}{(l+m+n)^{l+m+n}}\right]. \tag{18}$$