THEOREM 1. (IMPLICIT FUNCTION THEOREM, THEOREM 5.2.6 IN DR.RUNDE'S NOTES) *Let* $\varnothing \neq U \subset \mathbb{R}^{M+N}$ *be open, let* $F \in C^1(U, \mathbb{R}^N)$ *where* F *is a function of* (x, y) *with* $x \in \mathbb{R}^M, y \in \mathbb{R}^N$*, and let* $(x_0, y_0) \in U$ *be such that* $F(x_0, y_0) = 0$ *and* $\det \frac{\partial F}{\partial y}(x_0, y_0) \neq 0$*. Then there are neighborhoods* $V \subset \mathbb{R}^M$ *of* x_0 *and* $W \subset \mathbb{R}^N$ *of* y_0 *with* $V \times W \subset U$ *and a unique* $\phi \in C^1(V, \mathbb{R}^N)$ *such that*

i. $\phi(x_0) = y_0$ *and*

ii. $f(x, y) = 0$ *if and only if* $\phi(x) = y$ *for all* $(x, y) \in V \times W$*.*

THEOREM 2. (INVERSE FUNCTION THEOREM, THEOREM 5.2.5 IN DR. RUNDE'S NOTES) *Let* $\varnothing \neq U \subset \mathbb{R}^N$ *be open, let* $f \in C^1(U, \mathbb{R}^N)$*, and let* $x_0 \in U$ *be such that* $\det J_f(x_0) \neq 0$*. Then there is an open neighborhood* $V \subset U$ *of* x_0 *such that* f *is injective on* V*,* $f(V)$ *is open, and* $f^{-1}: f(V) \mapsto \mathbb{R}^N$ *is a* C^1*-function such that* $J_{f^{-1}} = J_f^{-1}$*.*

Remark 3. The two theorems are equivalent. We can prove either one and the other will follow.

- Implicit to inverse.
  We define $F(x, y) = f(x) - y$. Then we see that $\frac{\partial F}{\partial x} = J_f(x)$. Application of the implicit function theorem yields a function $X(y)$ such that $F(X(y), y) = 0$, in other words $f(X(y)) = y$.[1]

- Inverse to implicit.
  We define $F(x, y) = \begin{pmatrix} x \\ f(x, y) \end{pmatrix}$. It can be shown that $J_F(x, y) = \begin{pmatrix} I & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$ from which it follows that

$$\det J_F(x_0, y_0) = \det I \cdot \det\left(\frac{\partial f}{\partial y}(x_0, y_0)\right) \neq 0. \tag{1}$$

  Application of the inverse function theorem we have a function $G(x, y) = \begin{pmatrix} G_1(x, y) \\ G_2(x, y) \end{pmatrix}$ where $G_1: \mathbb{R}^{M+N} \mapsto \mathbb{R}^M$ and $G_2: \mathbb{R}^{M+N} \mapsto \mathbb{R}^N$ such that $G(F(x, y)) = \begin{pmatrix} x \\ y \end{pmatrix}$ and $F(G(u, v)) = \begin{pmatrix} u \\ v \end{pmatrix}$. Now define $Y(x) = G_2(x, 0)$. We have

$$\begin{pmatrix} x \\ f(x, Y(x)) \end{pmatrix} = F(x, Y(x)) = F(x, G_2(x, 0)). \tag{2}$$

  Now notice that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} G_1(F(x, y)) \\ G_2(F(x, y)) \end{pmatrix} = \begin{pmatrix} G_1(x, f(x, y)) \\ G_2(x, f(x, y)) \end{pmatrix} \tag{3}$$

  for every (x, y) in a certain neighborhood of (x_0, y_0). As $F \in C^1$ with $J_F(x_0, y_0)$ non-singular, F maps this neighborhood onto some neighborhood of $(x_0, 0) = F(x_0, y_0)$. Consequently for fixed x, $f(x, y)$ is onto some neighborhood of 0 and we can conlude that $G_1(u, v) = u$ in a neighborhood of $(x_0, 0)$.

  Now (2) becomes

$$\begin{pmatrix} x \\ f(x, Y(x)) \end{pmatrix} = F(x, G_2(x, 0)) = F(G(x, 0)) = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{4}$$

  and we have $f(x, Y(x)) = 0$ as desired.

IMPLICIT DIFFERENTIATION

Examples

We show through examples how to calculate derivatives of implicit/inverse functions.

One dependent variable

Example 4. Let $y = Y(x)$ be defined through $Y(1) = 1$ and

$$x^2 y^2 - 3y + 2x^3 = 0. \tag{5}$$

1. To see that $X(f(x)) = x$ we need that fact that f is 1-1 which is guaranteed by the condition $\det J_f(x_0) \neq 0$.

Find $Y'(1)$.

Solution. Taking $\frac{\mathrm{d}}{\mathrm{d}x}$ to

$$x^2 Y(x)^2 - 3Y(x) + 2x^3 = 0 \tag{6}$$

we have

$$2xY(x)^2 + 2x^2 Y(x) Y'(x) - 3Y'(x) + 6x^2 = 0. \tag{7}$$

Setting $x = 1$ we have

$$2 + 2Y'(1) - 3Y'(1) + 6 = 0 \Longrightarrow Y'(1) = 8. \tag{8}$$

Remark 5. In general, for the function Y defined by $f(x, y) = 0$, where $f: \mathbb{R}^2 \mapsto \mathbb{R}$, there holds $Y'(x) = -\left(\frac{\partial f}{\partial x}(x, Y(x))\right) / \left(\frac{\partial f}{\partial y}(x, Y(x))\right)$.

Exercise 1. Prove this formula.

However, as can be seen in the above example, often it is simpler to use chain rule instead of trying to remember the formula.

Example 6. Let $z = Z(x, y)$ be defined through

$$\sin z - xyz = 0. \tag{9}$$

Find $\frac{\partial Z}{\partial x}, \frac{\partial Z}{\partial y}$.

Solution. Taing $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, we easily obtain

$$\frac{\partial z}{\partial x} = \frac{yz}{\cos z - xy}, \qquad \frac{\partial z}{\partial y} = \frac{xz}{\cos z - xy}. \tag{10}$$

Two or more dependent variables

Let $f(x, y)$: $\mathbb{R}^{M+N} \mapsto \mathbb{R}^N$, assume that $y = Y(x)$ is an implicit function defined through $f(x, y) = 0$. Differentiating $g(x) := f(x, Y(x)) = 0$ through the chain rule we have

$$0 = \frac{\partial g_i}{\partial x_j} = \frac{\partial f_i}{\partial x_j} + \sum_{k=1}^{N} \frac{\partial f_i}{\partial y_k} \frac{\partial Y_k}{\partial x_j}. \tag{11}$$

If we denote by $\left(\frac{\partial f}{\partial x}\right)$ the matrix whose (i, j) entry is $\frac{\partial f_i}{\partial x_j}$, and use similar notations for $\frac{\partial f_i}{\partial y_k}$ and $\frac{\partial Y_k}{\partial x_j}$, we have

$$0 = \left(\frac{\partial f}{\partial x}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial Y}{\partial x}\right) \Longrightarrow J_Y(x) = -\left(\frac{\partial f}{\partial y}\right)^{-1}\left(\frac{\partial f}{\partial x}\right). \tag{12}$$

Remark 7. For the general case where there are two or more independent variables, it is more convenient to remember (12).

Example 8. Consider the system

$$x_1 y_2 - 4x_2 + 2e^{y_1} + 3 = 0 \tag{13}$$

$$2x_1 - x_3 - 6y_1 + y_2 \cos y_1 = 0 \tag{14}$$

Calculate the Jacobian of the implicit function $\boldsymbol{Y}(\boldsymbol{x})$ at $x_1 = -1, x_2 = 1, x_3 = -1, y_1 = 0, y_2 = 1$.

Solution. Let

$$f(\boldsymbol{x}, \boldsymbol{y}) := \begin{pmatrix} x_1 y_2 - 4x_2 + 2e^{y_1} + 3 \\ 2x_1 - x_3 - 6y_1 + y_2 \cos y_1 \end{pmatrix}. \tag{15}$$

Then we have

$$\left(\frac{\partial f}{\partial \boldsymbol{x}}\right) = \begin{pmatrix} y_2 & -4 & 0 \\ 2 & 0 & -1 \end{pmatrix}, \qquad \left(\frac{\partial f}{\partial \boldsymbol{y}}\right) = \begin{pmatrix} 2e^{y_1} & x_1 \\ -6 - y_2 \sin y_1 & \cos y_1 \end{pmatrix}. \tag{16}$$

At the specified point we have

$$\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)=\left(\begin{array}{ccc}1 & -4 & 0\\ 2 & 0 & -1\end{array}\right), \qquad \left(\frac{\partial f}{\partial \boldsymbol{y}}\right)=\left(\begin{array}{cc}2 & -1\\ -6 & 1\end{array}\right). \tag{17}$$

We see that

$$J_Y(-1,1,-1)=\left(\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{x}}\right)=\frac{1}{4}\left(\begin{array}{ccc}3 & -4 & -1\\ 10 & -24 & -2\end{array}\right). \tag{18}$$

Example 9. Let $z=f(x,y),\ g(x,y)=0$. Calculate $\frac{\mathrm{d}z}{\mathrm{d}x}$.

Solution. Let

$$\boldsymbol{F}(x,y,z)=\left(\begin{array}{c}f(x,y)-z\\ g(x,y)\end{array}\right). \tag{19}$$

Then we have

$$\left(\frac{\partial \boldsymbol{F}}{\partial x}\right)=\left(\begin{array}{c}\frac{\partial f}{\partial x}\\ \frac{\partial g}{\partial x}\end{array}\right), \qquad \left(\frac{\partial \boldsymbol{F}}{\partial(y,z)}\right)=\left(\begin{array}{cc}\frac{\partial f}{\partial y} & -1\\ \frac{\partial g}{\partial y} & 0\end{array}\right) \tag{20}$$

which gives

$$\frac{\partial(Y,Z)}{\partial x}=-\left(\begin{array}{cc}\frac{\partial f}{\partial y} & -1\\ \frac{\partial g}{\partial y} & 0\end{array}\right)^{-1}\left(\begin{array}{c}\frac{\partial f}{\partial x}\\ \frac{\partial g}{\partial x}\end{array}\right)=-\frac{1}{\frac{\partial g}{\partial y}}\left(\begin{array}{cc}0 & 1\\ -\frac{\partial g}{\partial y} & \frac{\partial f}{\partial y}\end{array}\right)\left(\begin{array}{c}\frac{\partial f}{\partial x}\\ \frac{\partial g}{\partial x}\end{array}\right). \tag{21}$$

Finally we have

$$\frac{\mathrm{d}Z}{\mathrm{d}x}=\frac{1}{\frac{\partial g}{\partial y}}\det\frac{\partial(f,g)}{\partial(x,y)}. \tag{22}$$