# 1. Introduction

## 1.1. Implicit and inverse functions

- A implicit function is a function defined "implicitly" through an equation:
$$F(x,y)=0. \tag{1}$$
A special case is "inverse function":
$$G(y)=x. \tag{2}$$
That this is a special case can be seen by setting $F(x,y):=G(y)-x$.

- Recall that a function is a triplet $(f,A,B)$, or $f: A\mapsto B$, where for every $x\in A$, there corresponds a unique $y\in B$ which we denote by $f(x)$. Thus to show that $F(x,y)=0$ defines a function $y=Y(x)$ implicitly, we need to show the existence of such $A$, $B$, and a mapping $x\mapsto y=Y(x)$. The key here is to show the uniqueness of $y$, that is
$$\forall x\in A, \qquad \text{there is a unique } y\in B \text{ such that } F(x,y)=0. \tag{3}$$

- In calculus, we study functions through its derivatives. Thus a follow-up questions is, if (3) is satisfied, and indeed $y$ is a function of $x$, that is $y=Y(x)$,
  - how regular is $Y$? Can we deduce its regularity from that of $F(x,y)$?
  - if $Y$ has up to $n$th order derivatives, can we calculate them without explicitly finding the formula of $Y$?

- In this unit we will answer these questions. In particular we will prove the following two theorems.

Theorem 1. (Implicit function theorem, Theorem 5.2.6 in Dr.Runde's notes) *Let $\varnothing\neq U\subset\mathbb{R}^{M+N}$ be open, let $F\in C^1(U,\mathbb{R}^N)$ where $F$ is a function of $(x,y)$ with $x\in\mathbb{R}^M$, $y\in\mathbb{R}^N$, and let $(x_0,y_0)\in U$ be such that $F(x_0,y_0)=0$ and $\det\frac{\partial F}{\partial y}(x_0,y_0)\neq 0$. Then there are neighborhoods $V\subset\mathbb{R}^M$ of $x_0$ and $W\subset\mathbb{R}^N$ of $y_0$ with $V\times W\subset U$ and a unique $\phi\in C^1(V,\mathbb{R}^N)$ such that*

*i. $\phi(x_0)=y_0$ and*

*ii. $f(x,y)=0$ if and only if $\phi(x)=y$ for all $(x,y)\in V\times W$.*

Theorem 2. (Inverse function theorem, Theorem 5.2.5 in Dr. Runde's notes) *Let $\varnothing\neq U\subset\mathbb{R}^N$ be open, let $f\in C^1(U,\mathbb{R}^N)$, and let $x_0\in U$ be such that $\det J_f(x_0)\neq 0$. Then there is an open neighborhood $V\subset U$ of $x_0$ such that $f$ is injective on $V$, $f(V)$ is open, and $f^{-1}: f(V)\mapsto\mathbb{R}^N$ is a $C^1$-function such that $J_{f^{-1}}=J_f^{-1}$.*

**Remark 3.** The two theorems are equivalent. We can prove either one and the other will follow.

## 1.2. Examples

- The inverse/implicit functions are defined locally.

**Example 4.** $f(x,y)=x^2+y^2-1$.
We see that
$$\frac{\partial f}{\partial y}(x,y)=2\,y. \tag{4}$$
Therefore if $(x_0,y_0)\neq(\pm 1,0)$, there are neighborhoods of $(x_0,y_0)$ in which we can represent $y=Y(x)$.
On the other hand, we see that at $(\pm 1,0)$, this is not possible.

- Why (and how) is a locally defined function useful?

**Example 5.** Consider the equation
$$y^3+x^2\,y^2-x\,y+x^4=0. \tag{5}$$

We see that when $x = 0$, $y = 0$. Thus it is natural to search for an implicit function $y = Y(x)$ such that $Y(0) = 0$ and when $x$ is close to 0, $(x, Y(x))$ satisfies (5). It turns out that $Y(x) \approx x^3$ near $x = 0$.

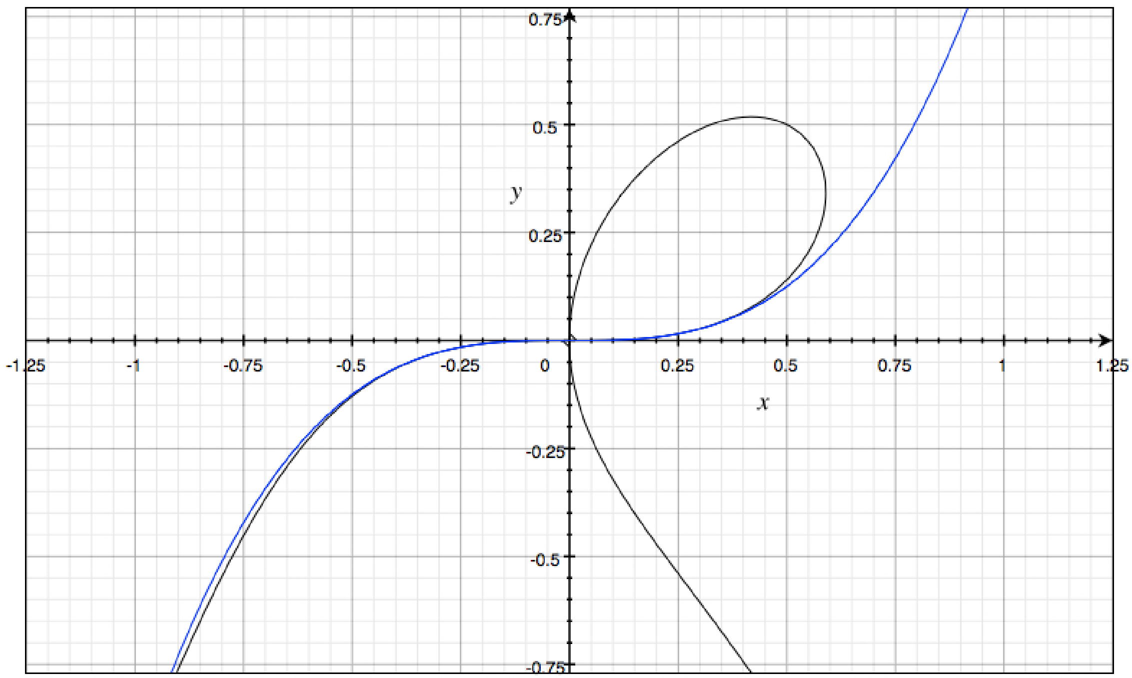


**Figure 1.** $y^3 + x^2 y^2 - x y + x^4 = 0$ v.s. $y = x^3$.

- The necessity implicitly defined functions.

**Example 6.** When studying planet orbits, Johannes Kepler tried to solve the following problem:

> Assume that the planet is moving according to Kepler's law, how can we find its position as a function of time?

We set up the problem in Figure 2 (Forgive my poor drawing! Texmacs does not have ellipsis drawing tool). $B$ is the position of say the moon, and $F$ is the position of the earth. The (half) ellipsis is the trajectory of $B$, which is an ellipsis with semi-axes $a$ and $b$. We draw an auxiliary half circle that is tangent to the ellipsis at the perigee $P$ and the apogee $A$. We then "project" $B$ to $B'$ on this half circle so that $B' B \perp A P$.

Let $O$ be the center of the half circle. We denote by $E$ the angle $\angle B' O P$, and call it the "eccentric anomaly". By Kepler's third law, the change rate of the area of the curvilinear triangle $BPF$ is constant, and can be easily calculated. We call this area the "mean anomaly" and denote it by $M$. We also denote by $e$ the eccentricity of the ellipsis, that is $|OF| = a\, e$. The task now is to obtain a formula for $E$ from the easy-to-obtain formula of $M$, that is writing $E$ as a function of $M$.

It is clear that the area of $OB'P$ is given by $\frac{E}{2} a^2$. Therefore the area of $OBP$ is $\frac{E}{2} a\, b$.

**Exercise 1.** Prove this.

Next we have

$$|B'' B| = \frac{b}{a} |B' B| = \frac{b}{a} a \sin E = b \sin E. \tag{6}$$

Therefore the area of the triangle $OBF$ is

$$\frac{1}{2} |OF| \cdot |BB''| = \frac{e\, a}{2} b \sin E = \frac{e}{2} a\, b \sin E. \tag{7}$$

Therefore the area of $FBP$ is given by

$$|OBP| - |OBF| = \frac{a\,b}{2}\,(E - e\sin E). \tag{8}$$

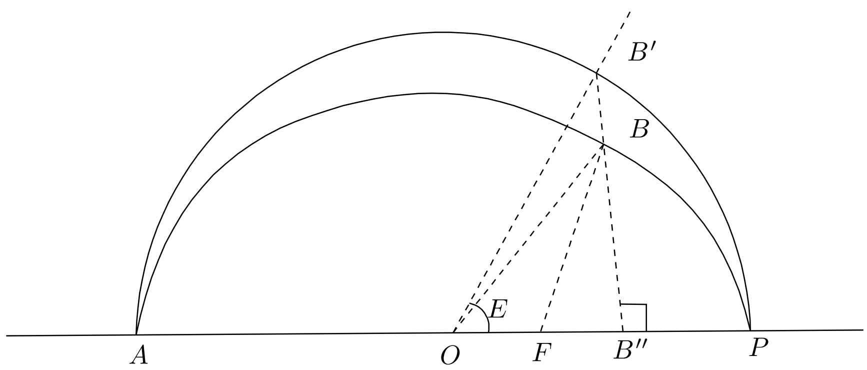


**Figure 2.** Derivation of Kepler's equation

Thus if we ''normalize'' $M$ by the factor $\frac{a\,b}{2}$, we reach the following relation between the known "mean anomaly" $M$ and the unknown "eccentric anomaly" $E$.

$$E = M + e\sin(E) \tag{9}$$

We see that $E$ as a function of $M$ is defined implicitly, and furthermore we note that it is not possible to solve it through explicit formula.

On the other hand, it is possible to solve $E$ through an "infinite series" formula, as we will see in a later lecture.

- Counter-examples.

**Example 7.** Let $f(x, y) = (y - x)^2$. Then we have $\frac{\partial f}{\partial y}(0, 0) = 0$. However it is clear that a smooth implicit function is defined through $f(x, y) = 0$.

**Example 8.** Let $f(x) := \begin{cases} \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$. It is easy to see that $f'(0) > 0$ but we now show that there does not exist an inverse function in any neighborhood of $x = 0$.

To do this we calculate

$$f'(x) = \frac{1}{2} - \cos\left(\frac{1}{x}\right) + 2\,x\sin\left(\frac{1}{x}\right). \tag{10}$$

Consider $x \in \left(-\frac{1}{4}, \frac{1}{4}\right)$. For these $x$ there holds $\left|\frac{1}{2} + 2\,x\sin\left(\frac{1}{x}\right)\right| < 1$. On the other hand, we know that there are $x_n^1, x_n^2 \longrightarrow 0$ such that $\cos\left(\frac{1}{x_n^1}\right) = 1$, $\cos\left(\frac{1}{x_n^2}\right) = -1$. Consequently, for any $0 < \varepsilon < \frac{1}{4}$, $f'(x)$ takes both positive and negative values in $(-\varepsilon, \varepsilon)$, which means $f(x)$ is not monotone on $(-\varepsilon, \varepsilon)$ and consequently cannot have a inverse function near $x = 0$.

**Exercise 2.** Which assumption of the inverse function theorem is not satisfied in Example 8?