# Integration on $\mathbb{R}^N$

Please read §4.1, §4.2, §4.4 of Dr. Runde's notes.[1]

## Content theory (Jordan measure).

- A "measure" on $\mathbb{R}^N$ is a function with a collection of subsets of $\mathbb{R}^N$ as its domain, and with $\mathbb{R}$ as its target space. We denote this function by $\mu$.
- "Axioms".
  1. $\mu \geqslant 0$.
  2. $\mu([0,1]^N) = 1$.
  3. $\mu(A + x_0) = \mu(A)$ for any $x_0 \in \mathbb{R}^N$.
  4. $\mu(r\,A) = r^N \mu(A)$ for any $r \in \mathbb{R}$.
  5. $\mu(A) \leqslant \mu(B)$ if $A \subseteq B$.
  6. $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \varnothing$.
- The measure of $N$-dimensional intervals and their finite unions.
  - $\mu((0,1)^N) = 1$.
  - $\mu([a_1, b_1] \times \cdots \times [a_N, b_N]) = \mu((a_1, b_1) \times \cdots \times (a_N, b_N)) = \prod_{j=1}^N (b_j - a_j)$.
  - The measure of any finite union of $N$-dimensional intervals can be uniquely determined from the axioms, through dividing them into disjoint intervals.
- Outer and inner measures.
  - Let $\mathcal{S} := \{$subsets of $\mathbb{R}^N$ that is a finite union of intervals$\}$. We define the Jordan outer measure

$$\mu_{\text{out}}(A) := \inf_{B \in \mathcal{S}, B \supseteq A} \mu(B), \tag{1}$$

    and the Jordan inner measure

$$\mu_{\text{in}}(A) := \sup_{B \in \mathcal{S}, B \subseteq A} \mu(B). \tag{2}$$

- Jordan measurable sets.
  - We say $A \subseteq \mathbb{R}^N$ is Jordan measurable if $\mu_{\text{out}}(A) = \mu_{\text{in}}(A)$, and denote this common value by $\mu(A)$.
  - If a set $A$ satisfies $\mu_{\text{out}}(A) = 0$, then it is Jordan measurable with $\mu(A) = 0$.

    **Exercise 1.** Prove that the Cantor set is Jordan measurable with $\mu(A) = 0$.

    **Exercise 2.** Let $A_1$, ..., $A_m$ be such that $\mu(A_i) = 0$, $i = 1, 2, ..., m$. Prove that $A := \cup_{i=1}^m A_i$ is Jordan measurable and furthermore $\mu(A) = 0$.

    **Exercise 3.** Find a sequence of sets $A_1, A_2, ...$ such that $\mu(A_i) = 0$ for every $i$ but for $A := \cup_{i=1}^\infty A_i$ there cannot hold $\mu(A) = 0$.

  - We have, for every $A \subseteq \mathbb{R}^N$,

$$\mu_{\text{out}}(A) = \mu_{\text{out}}(\bar{A}), \qquad \mu_{\text{in}}(A) = \mu_{\text{in}}(A^o). \tag{3}$$

  - We have

$$A \text{ is Jordan measurable} \Longleftrightarrow \mu(\partial A) = 0. \tag{4}$$

    **Exercise 4.** Prove that $\mathbb{Q} \cap [0,1]$ is not Jordan measurable.

  - If $A_1, A_2$ are Jordan measurable, so are $A_1 \cap A_2$, $A_1 \cup A_2$, $A_1 \backslash A_2$, $A_2 \backslash A_1$.

1. We will review §4.3, §4.5 later during the semester when the materials there are needed.

## Integration of functions.

- Axioms.

  "Integration" is a function from the $\mathcal{M} \times \mathcal{F}$ to $\mathbb{R}$: $I(\Omega, f) \in \mathbb{R}$. Here $\mathcal{M}$ is the collection of all measurable sets, and $\mathcal{F}$ is the collection of all functions.

  1. $I(\Omega, \mathbf{1}_A) = \mu(\Omega \cap A)$;
  2. $I$ is linear in $f$;
  3. $I(\Omega_1 \cup \Omega_2, f) = I(\Omega_1, f) + I(\Omega_2, f)$ if $\Omega_1 \cap \Omega_2 = \varnothing$;
  4. $I(\Omega, f_1) \leqslant I(\Omega, f_2)$ if $f_1 \leqslant f_2$.

- Simple functions.

  $f: \mathbb{R}^N \mapsto \mathbb{R}$ is simple if

  $$f(x) = \sum_{i=1}^{m} a_i \mathbf{1}_{A_i} \tag{5}$$

  where $A_1, \ldots, A_m$ are Jordan measurable, and $a_i \in \mathbb{R}$.

  **Exercise 5.** Prove that simple functions are integrable with $I(\Omega, f) = \sum_{i=1}^{m} a_i \mu(\Omega \cap A_i)$.

- Upper and lower integrals.

  Let $\mathcal{S}$ be the set of all simple functions. For an arbitrary $f: \mathbb{R}^N \mapsto \mathbb{R}$ we can define its upper integral

  $$I_{\text{up}}(\Omega, f) := \inf_{g \in \mathcal{S}, g \geqslant f} I(\Omega, g) \tag{6}$$

  and lower integral

  $$I_{\text{lo}}(\Omega, f) := \sup_{g \in \mathcal{S}, g \leqslant f} I(\Omega, g). \tag{7}$$

- Integrable functions.

  $f$ is integrable on $\Omega$ if

  $$I_{\text{up}}(\Omega, f) = I_{\text{lo}}(\Omega, f). \tag{8}$$

- Checking integrability.

  - $f \geqslant 0$ is integrable if and only if

    $$A := \{(x, y) \in \mathbb{R}^{N+1} | \, 0 \leqslant y \leqslant f(x), x \in \Omega\} \tag{9}$$

    is Jordan measurable in $\mathbb{R}^{N+1}$. Furthermore

    $$I(\Omega, f) = \mu_{N+1}(A). \tag{10}$$

  - $f$ is integrable if its graph has zero Jordan measure (content) in $\mathbb{R}^{N+1}$.
  - In particular, continuous functions are integrable.

## Some theoretical issues

### Fundamental theorem of calculus

- Recall that there are now two definition of "integration" for $N = 1$.
  - Indefinite integral: The solutions to the equation $y'(x) = f(x)$;
  - Riemann integral: The measure to the set $A := \{(x, y) | \, 0 \leqslant y \leqslant f(x), x \in \Omega\}$.
- They are related through the fundamental theorem of calculus:

  THEOREM. *Let $f: [a, b] \mapsto \mathbb{R}$ be "nice". Then*

  $$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a) \tag{11}$$

  *where the left hand side is Riemann integral, and $F(x)$ is one of the indefinite integrals of $f$.*

**Remark 1.** Continuous functios are "nice".

**Remark 2.** The situation becomes complicated when $f$ is not continuous. For example one can find a function $f$ that is not Riemann integrable but has an anti-derivative $F$. For (much) more on such issues, see the book "The Calculus Integral" by Brian S. Thomson, freely available at http://classicalrealanalysis.info/documents/T-CalculusIntegral-AllChapters-Portrait.pdf.

**Fubini**

- Fubini-type result:

THEOREM. (GENERIC FUBINI-TYPE) *If $f(x, y)$ is "nice", then*

$$\int_{\Omega_1 \times \Omega_2} f(x, y)\,\mathrm{d}(x, y) = \int_{\Omega_1} \left[\int_{\Omega_2} f(x, y)\,\mathrm{d}y\right] \mathrm{d}x = \int_{\Omega_2} \left[\int_{\Omega_1} f(x, y)\,\mathrm{d}x\right] \mathrm{d}y. \tag{12}$$

  - Be sure to understand the issues involved in a Fubini-type result:
    - i. Existence of the five integrals involved;
    - ii. Equalities of the integrals.

**Remark 3.** Continuous functions are "nice".

**Remark 4.** If $f$ satisfies the following, then it is "nice":

1. $f(x, y)$ is integrable on $\Omega_1 \times \Omega_2$;
2. For every $x$, $f(x, y)$ is integrable on $\Omega_2$;
3. For every $y$, $f(x, y)$ is integrable on $\Omega_1$.

- Understanding Fubini through upper and lower integrals.

$$I_{\text{up}}(\Omega_1, I_{\text{up}}(\Omega_2, f(\cdot, y))) \leqslant I_{\text{up}}(\Omega_1 \times \Omega_2, f(x, y)); \tag{13}$$

$$I_{\text{lo}}(\Omega_1, I_{\text{lo}}(\Omega_2, f(\cdot, y))) \geqslant I_{\text{lo}}(\Omega_1 \times \Omega_2, f(x, y)). \tag{14}$$

- Cavalieri's principle (Zu Geng principle).

If every slice of the two volumes has the same area, then the two volumes are equal.

**Example 5.** Archimedes' and Liu Hui/Zu Chongzhi/Zu Geng's proofs of the volume of the ball.[2]

2. See http://www.wlsh.tyc.edu.tw/ezfiles/2/1002/img/31/05OrCM_sphere.pdf. The article is in Chinese but the pictures should be quite self-explanatory.