1. Topology of $\mathbb{R}^N$.

- Open and closed balls.

$$B_r(x_0) := \{x \in \mathbb{R}^N | \, \|x - x_0\| < r\}; \qquad B_r[x_0] := \{x \in \mathbb{R}^N | \, \|x - x_0\| \leqslant r\}. \tag{1}$$

- Open and closed sets.

  - A set is open if and only if it is a union of open balls;

    Exercise 1. Prove that a subset of $\mathbb{R}^N$ is open if and only if it is a countable union of open balls.

  - A set is closed if and only if its complement is open.

  - A set A is closed if and only if $\{x_n\} \subseteq A$, $x_n \longrightarrow x$, then $x \in A$.

- Some new operations on sets.

  - Interior.

$$A^o := \cup_B \{B \subseteq A | \, B \text{ is open}\}. \tag{2}$$

  - Closure.

$$\bar{A} := \cap_B \{B \supseteq A | \, B \text{ is closed}\}. \tag{3}$$

  - Boundary.

$$\partial A := \bar{A} \backslash A^o. \tag{4}$$

Exercise 2. Let $A := \{(x, y) | \, x \in \mathbb{Q}, y \in \mathbb{Q}\}$ and $B := \left\{ \left(\frac{1}{n}, \frac{1}{m} \right) | \, n, m \in \mathbb{N} \right\}$. Calculate their interior, closuer, and boundary.

Exercise 3. Let $A, B \subseteq \mathbb{R}^N$ be arbitrary. Prove the following

$$(A^o)^o = A^o, \qquad (A \cap B)^o = A^o \cap B^o, \qquad \overline{(\bar{A})} = \bar{A}, \qquad \overline{(A \cup B)} = \bar{A} \cup \bar{B}, \tag{5}$$

$$\partial(\partial A) \subseteq \partial A, \qquad \partial(A \cup B) \subseteq (\partial A) \cup (\partial B), \qquad \partial(A \cap B) \supseteq (\partial A) \cap (\partial B). \tag{6}$$

Exercise 4. Prove that x is a cluster point of A if and only if $x \in \overline{A - \{x\}}$.

A fun problem. Let $A \subset \mathbb{R}^N$. Apply ${}^c, {}^o, {}^-$ to A finitely many times in any order you want. How many different set can you get?[1]

- Compactness.

  - Definition. A set $K \subset \mathbb{R}^N$ is compact if every open cover has a finite subcover.

    Exercise 5. Let $A = \{x_1, x_2, \ldots\} \cup \{y_1, y_2, \ldots\}$. Assume that both $\{x_n\}$ and $\{y_n\}$ are convergent. Prove or disprove: A is compact.

  - A set $K \subset \mathbb{R}^N$ is compact if and only if every sequence in K has a convergent subsequence

1. At most 14.

whose limit is in K.

Proof.

- If. Consider an arbitrary open covering of K. Clearly we can assume that it is a countable covering: $K \subseteq \cup_{i=1}^{\infty} U_i$. Denote $V_i := \cup_{j=1}^{i} U_j$. Ntoe that $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$. If there is a i such that $K \subseteq V_i$ then we have a finite subcover. Otherwise we have a sequence $\{x_n\} \subseteq K$ such that $x_n \notin V_n$. By assumption it has a convergent subsequence $x_{n_k} \longrightarrow x_0 \in K$. There is a $m_0 \in \mathbb{N}$ such that $x_0 \in V_{m_0}$. As V_{m_0} is open, there is $k_0 \in \mathbb{N}$ such that $x_{n_k} \in V_{m_0}$ for all $k > k_0$. Now again because $K \subseteq \cup_{i=1}^{\infty} V_i$, there are $m_1, \ldots, m_{k_0}$ such that $x_{n_1} \in V_{m_1}, \ldots, x_{n_{k_0}} \in V_{m_{k_0}}$. Finally let $M := \max\{m_0, m_1, \ldots, m_{k_0}\}$ we see that $x_{n_k} \in V_M$ for all $k \in \mathbb{N}$. Contradiction.

- Only if. Let K be compact. Let $\{x_n\}$ be an arbitrary sequence in K. Assume the contrary, that is $\{x_n\}$ does not have any convergent subsequence whose limit is in K. Thus we can assume that $x_i \neq x_j$ whenever $i \neq j$.

  Under such assumptions, for any $y \in K$, there is $r(y) > 0$ such that

  $$B(y) := B_{r(y)}(y) \cap \{x_n\} \subseteq \{y\}. \tag{7}$$

  Clearly $K \subseteq B(y)$. By compactness of K there is a finite sub-cover:

  $$K \subseteq B_{r(y_1)}(y_1) \cup \cdots \cup B_{r(y_k)}(y_k). \tag{8}$$

  In particular

  $$\{x_n\} \subseteq B_{r(y_1)}(y_1) \cup \cdots \cup B_{r(y_k)}(y_k). \tag{9}$$

  Now by (7) there holds $\{x_n\} \subseteq \{y_1, \ldots, y_k\}$. Contradiction. □

  - Heine-Borel. A set $K \subset \mathbb{R}^N$ is compact if and only if it is both closed and bounded.

    Exercise 6. Prove that $K \subset \mathbb{R}^N$ is closed and bounded if and only if every sequence in K has a convergent subsequence whose limit is in K. Thus proving Heine-Borel.

- Connectness.

  - Definition. A set A is connected if and only if there do not exist open sets U, V such that $A \subseteq U \cup V$ and $U \cap V = \varnothing$.

    Exercise 7. Let $A := \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) | \, x > 0\right\} \cup \{(0, y) | -1 \leqslant y \leqslant 1\}$. Is A connected? Justify your answer.

Exercise 8. Let $A = A_1 \cup A_2$ where both A_1, A_2 are closed. Furthermore assume $A_1 \cap A_2 = \varnothing$. Prove or disprove: A is disconnected.

1. Limit and Continuity of Functions

Please read §2.2–2.4 of Dr. Runde's book.

1.1. Definitions and basic properties

- Let $f: \mathbb{R}^N \mapsto \mathbb{R}^M$. We say $\lim_{x \to x_0} f(x) = L$ if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \qquad 0 < \|x - x_0\| < \delta \Longrightarrow \|f(x) - L\| < \varepsilon. \tag{10}$$

  Note the $0 <$.

- When $\lim_{x \to x_0} f(x) = f(x_0)$, we say f is continuous at x_0.

Exercise 9. Let $g: \mathbb{R}^N \mapsto \mathbb{R}^M$, $f: \mathbb{R}^M \mapsto \mathbb{R}^K$. Prove or disprove the following statements:

a) If $\lim_{x \to x_0} g(x) = y_0$ and $\lim_{y \to y_0} f(y) = L$, then $\lim_{x \to x_0} f(g(x)) = L$;

b) If $g(x)$ is continuous at x_0 and $f(y)$ is continuous at $y_0 := g(x_0)$, then $f(g(x))$ is continuous at x_0.

- Properties of continuous functions.
  - Reaches maximum and minimum over compact sets.
  - When $M = 1$ enjoys intermediate value property.
  - D compact, f continuous, then $f(D)$ compact;
  - D connected, f continuous, then $f(D)$ connected.
  - U open, f continuous, then $f^{-1}(U)$ open.

  Exercise 10. Let D be compact. Prove or disprove: $f^{-1}(D)$ is compact.

  Exercise 11. Let D be connected. Prove or disprove: $f^{-1}(D)$ is connected.

  Exercise 12. Let U be open. Prove or disprove: $f(U)$ is open.

1.2. Fine properties and pitfalls

- Directional limit.

Example 1. Consider $f(x, y) = \begin{cases} \frac{x\,y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$. Then we see that

$$\lim_{y \longrightarrow 0} \left[\lim_{x \longrightarrow 0} f(x, y) \right] = \lim_{x \longrightarrow 0} \left[\lim_{x \longrightarrow 0} f(x, y) \right] = 0 \tag{11}$$

and furthermore $\lim_{(x, y) \longrightarrow 0 \text{ along a straight line}} f(x, y) = 0$, but

$$\lim_{(x, y) \longrightarrow (0, 0)} f(x, y) \tag{12}$$

does not exist.

Exercise 13. Let $f: \mathbb{R}^2 \mapsto \mathbb{R}$ be such that $\lim_{t \longrightarrow 0} f(x(t), y(t)) \longrightarrow 0$ for every smooth curve $(x(t), y(t))$ satisfying $\lim_{t \longrightarrow 0} (x(t), y(t)) = (0, 0)$. Prove or disprove: $\lim_{(x, y) \longrightarrow (0, 0)} f(x, y) = 0$.

Example 2. Consider $f(x, y) = (x + y) \sin\left(\frac{1}{x}\right) \sin\left(\frac{1}{y}\right)$ with domain $\{(x, y) |\, x \neq 0, y \neq 0\}$. Then we have

$$\lim_{(x, y) \longrightarrow (0, 0)} f(x, y) = 0 \tag{13}$$

but

$$\text{neither } \lim_{y \longrightarrow 0} \left[\lim_{x \longrightarrow 0} f(x, y) \right] \text{ nor } \lim_{x \longrightarrow 0} \left[\lim_{x \longrightarrow 0} f(x, y) \right] \tag{14}$$

- Darboux functions.
  - A function that has the intermediate value property is called a "Darboux function".

Example 3. Let $f(x) := \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then $f(x)$ is Darboux but not continuous.

Remark 4. It is possible to construct a function that is Darboux but is nowhere continuous. An example is "Conway's base 13 function" which takes every real value in every interval. A very accessible explanation is available at https://en.wikipedia.org/wiki/Conway_base_13_function.

Exercise 14. Given the above, prove that Conway's base 13 function is nowhere continuous.