As the domains and targets of functions in calculus are both subsets of $\mathbb{R}^N = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times), we need a better understanding of $\mathbb{R}^N$.

Note. Please read §1.3–§2.1 of Dr. Runde's notes.

1 Geometry of $\mathbb{R}^N$.

1.1 $N = 1$.

- The definition of $\mathbb{R}$ follows the following steps:

  1. Define $\mathbb{N}$;
  2. Define $\mathbb{Q}^+$ from $\mathbb{N} \times \mathbb{N}$;
  3. Great Leap Forward!! Define $\mathbb{R}^+$ from $\mathbb{Q}^+$;
  4. Define $\mathbb{R}$ from $\mathbb{R}^+$.

  For more details, see my lecture notes of Week 13 of Fall 2013 Math 217.

- Key properties of $\mathbb{R}$:

  1. It is a field, that is you can $+, -, \times, \div$ in it freely (except for $\div 0$);
  2. It is ordered;
  3. It is not countable;
  4. It has the least upper bound property, that is $\sup A \in \mathbb{R}$ for any $A \subset \mathbb{R}$ that is bounded above.

     Exercise 1. Prove that the least upper bound property is equivalent to completeness for $\mathbb{R}$.

1.2 $N > 1$.

Key differences between $\mathbb{R}$ and $\mathbb{R}^N$

1. It is not a field (unless $N = 2$);
2. It cannot be ordered naturally and conveniently: There is no continuous bijection from $\mathbb{R}$ to $\mathbb{R}^N$.[1]
3. There are some new operations:

   - Inner product:

   $$x \cdot y := x_1 y_1 + \cdots + x_N y_N. \tag{1}$$

   - Norm:

   $$\|x\| := \sqrt{x \cdot x}. \tag{2}$$

1. The situation changes when we drop either "bijection" or "continuous". -

- Key properties of inner product/norm:

  i. Cauchy-Schwarz:

$$|x \cdot y| \leqslant \|x\| \, \|y\|. \tag{3}$$

Exercise 2. Prove Cauchy-Schwarz through exploring the following fact: $\|x + t\, y\| \geqslant 0$ for all $t \in \mathbb{R}$.

Exercise 3. Prove Cauchy-Schwarz by writing $\|x\|^2 \, \|y\|^2 - |x \cdot y|^2$ into a square.

  ii. Triangle inequality:

$$\|x + y\| \leqslant \|x\| + \|y\|. \tag{4}$$

Exercise 4. Prove triangle inequality using Cauchy-Schwarz.

- Cross product $(N = 3)$:

$$x \times y := (x_2\, y_3 - y_2\, x_3, x_3\, y_1 - x_1\, y_3, x_1\, y_2 - x_2\, y_1). \tag{5}$$

- Determinant: There is a unique function

$$\det\colon \mathbb{R}^N \times \cdots \times \mathbb{R}^N \mapsto \mathbb{R} \tag{6}$$

  where the Cartesian product involves N $\mathbb{R}^N$'s, satisfying

  a. det is linear in each of its N variables;

  b. $\det(\cdots, x, \cdots, y, \cdots) = -\det(\cdots, y, \cdots, x, \cdots)$;

  c. $\det(e_1, \ldots, e_N) = 1$ where e_i is the vector with ith component 1 and others 0.

Limits of sequences in $\mathbb{R}^N$

- Recall the definition of limit of sequences in $\mathbb{R}$:

$$\lim_{n \to \infty} x_n = x \quad \text{if and only if} \quad \forall \varepsilon > 0 \; \exists N \in \mathbb{N}, \qquad n > N \Longrightarrow |x_n - x| < \varepsilon. \tag{7}$$

- The generalization to $\mathbb{R}^N$ is immediate after identifying the norm as the counter-part of the absolute value.

- Note that if $x^{(n)} := \left(x_1^{(n)}, \ldots, x_N^{(n)}\right)$, then $x^{(n)} \longrightarrow x^{(0)}$ is equivalent to $x_i^{(n)} \longrightarrow x_i^{(0)}$ for all $i = 1, 2, \ldots, N$.

- Almost every result about sequence convergence in $\mathbb{R}$ can be generalized to $\mathbb{R}^N$.

  - Cauchy sequence.

  - Bolzano-Weierstrass.

Theorem 1. *Every bounded, infinite subset $S \subset \mathbb{R}^N$ has a cluster point.*

Exercise 5. Critique the following "proof" of Bolzano-Weierstrass. If you think it is not correct, can you fix it?

Proof. We apply induction on N.

- $N=1$. In this case let $a_0 := \inf S$. If $a_0 \notin S$, then there must be a decreasing sequence $\{x_n\} \subset S$ such that $x_n \searrow a_0$, which means a_0 is the desired cluster point. On the other hand, if $a_0 \in S$, we denote it by x_0.

  Now consider $S_1 := S - \{x_0\}$, and $a_1 := \inf S_1$. If $a_1 \notin S_1$ then it is a cluster point of S. Otherwise denote $x_1 := a_1$. Note that $x_1 > x_0$.

  Repeat the above. If at one step we have $a_k \notin S_k$, then a_k is a cluster point. If there is no such k, then we have a sequence $\{x_n\} \subset S$ that is strictly increasing and bounded above. The limit of $\{x_n\}$ now is the desired cluster point.

- Assume that the theorem has been proved for $\mathbb{R}^N$. Consider $S \subset \mathbb{R}^{N+1}$ that is bounded and infinite. Let $T \subset \mathbb{R}$ be defined as

  $$T := \{x \in \mathbb{R} | \, (x_1, \ldots, x_N, x) \in S \text{ for some } x_1, \ldots, x_N\}. \tag{8}$$

  Then by the $N=1$ case there is a sequence in T converging to some $x_{N+1}^{(0)}$. Now denote this sequence by $\{x^{(n)}\}$ where $x^{(n)} := \left(x_1^{(n)}, \ldots, x_{N+1}^{(n)}\right)$. We have just shown that $x_{N+1}^{(n)} \longrightarrow x_{N+1}^{(0)}$. By the induction hypothesis there is a subsequence

  $$\left(x_1^{(n_k)}, \ldots, x_N^{(n_k)}\right) \longrightarrow \left(x_1^{(0)}, \ldots, x_N^{(0)}\right). \tag{9}$$

  Now $x^{(n_k)} \longrightarrow \left(x_1^{(0)}, \ldots, x_{N+1}^{(0)}\right)$ and the proof is finished. □

- Cluster point v.s. limit.

  Exercise 6. Prove or disprove:

  $$x_0 \text{ is a cluster point of a set } A \Longleftrightarrow \text{There is a sequence } \{x_n\} \subset A \text{ such that } x_n \longrightarrow x_0. \tag{10}$$

  Exercise 7. Is there any difference between Bolzano-Weierstrass and the following statement: Any bounded sequence in $\mathbb{R}^N$ has a convergent subsequence?