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FUNCTIONS

- We have seen that functions play a central role in calculus. In fact we may say that calculus is the study of a special class of functions.
- The modern definition of a function.

  A triplet: (f, A, B) where A, B are two sets and f is a subset of $A \times B$, the set of all ordered pairs (a, b) with $a \in A, b \in B$. It is required that f satisfy the following condition: for each $x \in A$, there is a unique $y \in B$ such that $(x, y) \in f$. For historic reasons, most of the times this is written as $y = f(x)$. We also write (f, A, B) as $f: A \mapsto B$.
- Some terminologies.
  - For a function $f: A \mapsto B$, we call A the domain of f, and B the target of f.
  - For $X \subseteq A$, we can define its image:
$$f(X) := \{f(x) | x \in X\}; \tag{1}$$
  - For $Y \subseteq B$, we can define its inverse image:
$$f^{-1}(Y) := \{x \in A | f(x) \in Y\}. \tag{2}$$
  - We say f is
    - injective if $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$. Note that this is equivalent to $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$;
    - surjective if $f(A) = B$;
    - bijective if it is both injective and surjective.

  Exercise 1. Let $f: A \mapsto B$ be a fucntion. Let $X_1, X_2 \subseteq A$ and $Y_1, Y_2 \subseteq B$.

  a) Prove: If $X_1 \subseteq X_2$, then $f(X_1) \subseteq f(X_2)$;

  b) Prove: If $Y_1 \subseteq Y_2$, then $f^{-1}(Y_1) \subseteq f^{-1}(Y_2)$;

  c) Is it true that $X_1 \subsetneq X_2$ implies $f(X_1) \subsetneq f(X_2)$? Justify your answer;

  d) Is it true that $Y_1 \subsetneq Y_2$ implies $f^{-1}(Y_1) \subsetneq f^{-1}(Y_2)$? Justify your answer.

  Exercise 2. Let $X \subseteq A, Y \subseteq B$ and $f: A \mapsto B$. Prove that

  a) $f(f^{-1}(Y)) \subseteq Y$;

  b) $f^{-1}(f(X)) \supseteq X$;

  c) If $Y \subseteq f(A)$, then $f(f^{-1}(Y)) = Y$.
- Inverse function.
  - When $f: A \mapsto B$ is bijective, the function $g \subset B \times A$ defined through
$$g := \{(y, x) | (x, y) \in f\} \tag{3}$$
    is also a function. We call $g: B \mapsto A$ the inverse function of $f: A \mapsto B$.
  - Do not confuse inverse function with inverse image!

  Exercise 3. Prove or disprove: $f: \mathbb{R} \mapsto \mathbb{R}$ has an inverse function g if and only if f is strictly increasing or strictly decreasing.
- Functions in calculus.
  - Functions studied in calculus are those $f: A \mapsto B$ with $A \subseteq \mathbb{R}^N$ for some N and $B \subseteq \mathbb{R}^M$ for some M.
  - $\mathbb{R}^N := \mathbb{R} \times \cdots \times \mathbb{R}$. $\mathbb{R}^1 := \mathbb{R}$.
- Some important examples of functions in calculus.
  - A nice function that does not looks so.
$$f(x) := \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}; \tag{4}$$

Exercise 4. Prove that $f(x)$ is infinitely differentiable.

Exercise 5. Calculate the Taylor expansion of f at $x=0$. What do you observe?

Exercise 6. Can you define a similar function on $\mathbb{R}^2$?

- A function with "monsters" hidden.

$$f(x) = x^k \sin\left(\frac{1}{x}\right). \tag{5}$$

Exercise 7. For what values of k is $f(x)$

a) defined on the whole of $\mathbb{R}$?

b) continuous on the whole of $\mathbb{R}$?

c) differentiable on the whole of $\mathbb{R}$?

d) twice differentiable on the whole of $\mathbb{R}$?

- A very bad function and its slightly better variant.

$$D(x) := \left\{ \begin{array}{ll} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{array} \right., \qquad R(x) := \left\{ \begin{array}{ll} \frac{1}{q} & x \in \mathbb{Q}, x = \frac{p}{q}, (p, q) = 1, q > 0 \\ 0 & x \notin \mathbb{Q} \end{array} \right.. \tag{6}$$

Exercise 8. Prove that

a) $D(x)$ is no where continuous, and is not Riemann integrable;

b) $R(x)$ is continuous at irrational numbers and discontinuous at rational numbers;

c) $R(x)$ is Riemann integrable.

- A bad function constructed from nice functions.

$$f(x) := \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}. \tag{7}$$

Exercise 9. Use a computer to plot $\sum_{n=1}^{N} \frac{\sin(n^2 x)}{n^2}$ for various N, say $N = 10, 20, 50.$[1]

Exercise 10. Looking at the plots, where do you think the function $f(x)$ could be differentiable?

Additive Functions.

Let $f: \mathbb{R} \mapsto \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$. Such f is called "additive". What can we say about such functions?

Exercise 11. Assume that f is continuous. Prove that $f(x) = a\,x$ for some $a \in \mathbb{R}$.

THEOREM. *If f is additive but not linear, then the graph of f is dense in $\mathbb{R}^2$.*

Proof. In this case there are $x_1 \neq x_2$ such that the two vectors $\left(\begin{smallmatrix} x_1 \\ f(x_1) \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x_2 \\ f(x_2) \end{smallmatrix}\right)$ are linearly independent in $\mathbb{R}^2$. Now if we think of f as a subset of $\mathbb{R}^2$, then additivity implies:

$$\left(\begin{array}{c} a \\ b \end{array}\right), \left(\begin{array}{c} a' \\ b' \end{array}\right) \in f \Longrightarrow \left(\begin{array}{c} a + a' \\ b + b' \end{array}\right) \in f. \tag{8}$$

From this we see that

$$q_1 \left(\begin{array}{c} x_1 \\ f(x_1) \end{array}\right) + q_2 \left(\begin{array}{c} x_2 \\ f(x_2) \end{array}\right) \in f, \qquad \forall q_1, q_2 \in \mathbb{Q} \tag{9}$$

and the conclusion follows. □

Remark 1. Such pathological functions can be constructed with the help of Axiom of Choice.

1. In OS X, the application "grapher" can do this very easily.