Math 317 Winter 2017 Homework 5 Solutions

Due Thursday Apr. 6, 2017 5pm

- The total points of this homework is 20.
- You need to fully justify your answer – for example, prove that your function indeed has the specified property – for each problem.

Question 1. (8 pts) *Calculate the explicit formulas of*

a) (4 pts) $f(x) := \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$,

b) (4 pts) $f(x) := \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$.

Justify your calculation.

Proof.

a) The radius of convergence is

$$R := \left(\limsup_{n \to \infty} \left(\frac{1}{2n+1} \right)^{1/(2n+1)} \right)^{-1} = 1. \tag{1}$$

Thus $f(x)$ is defined on $(-1,1)$. Further we notice that $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$ diverges at ± 1. So the domain of $f(x)$ is $(-1,1)$.

To calculate $f(x)$, we notice that for $x \in (-1,1)$, due to properties of power series,

$$f'(x) = \sum_{n=0}^{\infty} \left(\frac{x^{2n+1}}{2n+1} \right)' = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} = \frac{1}{2} \left[\frac{1}{1-x} + \frac{1}{1+x} \right]. \tag{2}$$

Therefore

$$f(x) = C + \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right|. \tag{3}$$

Setting $x=0$ in the power series we see that $f(0)=0$. Consequently $C=0$ and $f(x) = \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right|$.

b) The radius of convergence is

$$R := \left(\limsup_{n \to \infty} \left(\frac{1}{n(n+1)} \right)^{1/n} \right)^{-1} = 1. \tag{4}$$

Thus $f(x)$ is defined on $(-1,1)$. Further we notice that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ also converge so the domain of $f(x)$ is $[-1,1]$.

We first calculate $f(x)$ on $(-1,1)$. We have

$$x\, f(x) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} \tag{5}$$

and consequently

$$(x\, f(x))'' = \sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)}. \tag{6}$$

Therefore

$$(x\, f(x))' = C - \ln(1-x). \tag{7}$$

Note that as $x \in (-1,1)$ we do not need absolute value. Noticing that

$$(x\, f(x))'(0) = \sum_{n=1}^{\infty} \frac{x^n}{n} |_{x=0} = 0, \tag{8}$$

we have $C=0$ and $(x\, f(x))' = -\ln(1-x)$.

Integrating again we have

$$x\,f(x) = (1-x)\ln(1-x) + x + C. \tag{9}$$

As $x\,f(x) = 0$ at $x = 0$, we have $C = 0$. Consequently

$$f(x) = \frac{1-x}{x}\ln(1-x) + 1. \tag{10}$$ □

QUESTION 2. (8 PTS) *Let* $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 \leqslant x \leqslant \pi \end{cases}$ *and be* 2π*-periodic.*

a) (4 PTS) *Calculate the Fourier series expansion of* $f(x)$*;*

b) (4 PTS) *Use this expansion to prove*

$$\sum_{n\ odd} \frac{1}{n^2} = \frac{\pi}{8}. \tag{11}$$

Solution.

a) We have $L = \pi$. Calculate:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}x = \frac{\pi}{2}. \tag{12}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n\,x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\cos n\,x\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}\left(\frac{\sin n\,x}{n}\right) \\ &= -\frac{1}{\pi}\int_0^{\pi} \frac{\sin n\,x}{n}\,\mathrm{d}x \\ &= \frac{(-1)^n - 1}{\pi\,n^2}. \end{aligned} \tag{13}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n\,x\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin n\,x\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}\left(-\frac{\cos n\,x}{n}\right) \\ &= \frac{1}{\pi}\left[-\frac{\pi\,(-1)^n}{n} + \int_0^{\pi} \frac{\cos n\,x}{n}\,\mathrm{d}x\right] \\ &= \frac{(-1)^{n+1}}{n}. \end{aligned} \tag{14}$$

Therefore the Fourier series expansion is

$$f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n - 1}{\pi\,n^2}\cos n\,x + \frac{(-1)^{n+1}}{n}\sin n\,x\right\}. \tag{15}$$

b) The function this series converges to is a 2π-periodic function

$$\tilde{f}(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \\ \frac{\pi}{2} & x = \pi \end{cases}. \tag{16}$$

At $x = \pi$, we have

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi\,n^2}(-1)^n = \frac{\pi}{4} + \sum_{k=1}^{\infty}\frac{2}{\pi\,(2\,k+1)^2} \tag{17}$$

and the conclusion follows.

QUESTION 3. (4 PTS) *Let $D_N(t)$ be the Dirichlet kernel. Prove*

$$\lim_{N\longrightarrow\infty}\int_{-\pi}^{\pi}|D_N(t)|\,\mathrm{d}t=\infty. \tag{18}$$

Proof. We have

$$D_N(x)=\frac{\sin\frac{2\,N+1}{2}\,x}{2\,\pi\sin\frac{x}{2}}. \tag{19}$$

Therefore we have

$$\begin{aligned}
\pi\int_{-\pi}^{\pi}|D_N(x)|\,\mathrm{d}x &= \int_0^{\pi}\frac{\left|\sin\frac{2\,N+1}{2}\,x\right|}{\sin\frac{x}{2}}\,\mathrm{d}x\\
&> \int_0^{2\pi/(2N+1)}\frac{\left|\sin\frac{2\,N+1}{2}\,x\right|}{\sin\frac{x}{2}}\,\mathrm{d}x+\int_{\frac{2\pi}{2N+1}}^{\frac{4\pi}{2N+1}}\frac{\left|\sin\frac{2\,N+1}{2}\,x\right|}{\sin\frac{x}{2}}\,\mathrm{d}x\\
&\quad+\cdots+\int_{\frac{2(N-1)\pi}{2N+1}}^{\frac{2N\pi}{2N+1}}\frac{\left|\sin\frac{2\,N+1}{2}\,x\right|}{\sin\frac{x}{2}}\,\mathrm{d}x\\
&> \int_0^{\frac{2\pi}{2N+1}}\frac{\left|\sin\frac{2\,N+1}{2}\,x\right|}{\frac{\pi}{2\,N+1}}\,\mathrm{d}x+\cdots+\int_{\frac{2(N-1)\pi}{2N+1}}^{\frac{2N\pi}{2N+1}}\frac{\left|\sin\frac{2\,N+1}{2}\,x\right|}{\frac{N\pi}{2\,N+1}}\,\mathrm{d}x\\
&= \frac{2\,N+1}{\pi}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)\int_0^{\frac{2\pi}{2N+1}}\left|\sin\frac{2\,N+1}{2}\,x\right|\,\mathrm{d}x\\
&= \frac{2}{\pi}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)\int_0^{\pi}|\sin x|\,\mathrm{d}x\\
&= \frac{4}{\pi}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right).
\end{aligned} \tag{20}$$

In the above we have used the fact that $\left|\sin\frac{2\,N+1}{2}\,x\right|$ is periodic with period $\frac{2\,\pi}{2\,N+1}$. Now it's clear that

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}|D_N(x)|\,\mathrm{d}x=\infty. \tag{21}$$ □