# Math 317 Winter 2017 Homework 4 Solutions

Due Thursday Mar. 23, 2017 5pm

- The total points of this homework is 20.
- You need to fully justify your answer – for example, prove that your function indeed has the specified property – for each problem.

Question 1. (8 pts) *Let $f(x)$ be defined through*

$$\sum_{n=1}^{\infty} \frac{n-1}{n+1} \left( \frac{x}{3\,x+1} \right)^n. \tag{1}$$

*a)* (4 pts) *Find the domain $A$ of $f(x)$, that is find all $x \in \mathbb{R}$ such that the series converges.*

*b)* (4 pts) *Is the convergence uniform on $A$? Justify your claim.*

**Proof.**

a) We apply ratio test:

$$\frac{\left| \frac{n}{n+2} \left( \frac{x}{3\,x+1} \right)^{n+1} \right|}{\left| \frac{n-1}{n+1} \left( \frac{x}{3\,x+1} \right)^{n} \right|} = \frac{n\,(n+1)}{(n+2)\,(n-1)} \left| \frac{x}{3\,x+1} \right| \longrightarrow \left| \frac{x}{3\,x+1} \right|. \tag{2}$$

We consider three cases:

- $|x| < |3\,x+1| \Longleftrightarrow x < -\frac{1}{2}$ or $x > -\frac{1}{4}$. The series converges;
- $|x| > |3\,x+1| \Longleftrightarrow -\frac{1}{2} < x < -\frac{1}{4}$. The series diverges.
- $x = -\frac{1}{2}$ or $-\frac{1}{4}$. In this case $\left| \frac{n-1}{n+1} \left( \frac{x}{3\,x+1} \right)^n \right| = \left| \frac{n-1}{n+1} \right| \not\longrightarrow 0$ therefore the series diverges.

So the domain of $f$ is $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{4}, \infty\right)$.

b) The convergence is not uniform. We show that the series is not uniformly Cauchy. Let $N \in \mathbb{N}$ be arbitrary. Let $n > N$ be arbitrary. We have

$$\lim_{x \nearrow -\frac{1}{2}} \left| \frac{n-1}{n+1} \left( \frac{x}{3\,x+1} \right)^n \right| = \frac{n-1}{n+1} > \frac{1}{3}. \tag{3}$$

The conclusion now follows.

□

Question 2. (8 pts) *Consider the function defined through*

$$f(x) := \sum_{n=1}^{\infty} \frac{\sin(n\,x)}{n^2}. \tag{4}$$

*a)* (4 pts) *Find the domain $A$ of $f(x)$;*

*b)* (4 pts) *Prove or disprove: $f(x)$ is continuous on $A$.*

**Solution.**

a) We show that $f$ is defined for all $x \in \mathbb{R}$. We note that $\left| \frac{\sin(n\,x)}{n^2} \right| \leqslant \frac{1}{n^2}$ for all $x \in \mathbb{R}$. As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by Comparison $\sum_{n=1}^{\infty} \frac{\sin(n\,x)}{n^2}$ converges and $f(x)$ is defined for all $x \in \mathbb{R}$.

b) As $\left|\frac{\sin(n\,x)}{n^2}\right| \leqslant \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{\sin(n\,x)}{n^2}$ converges uniformly on $\mathbb{R}$ by Weierstrass' M-test. Since for each fixed $n$, $\frac{\sin(n\,x)}{n^2}$ is continuous, $f(x)$ is also continuous.

QUESTION 3. (4 PTS) *Let $f_n(x)$ be continuous on $[a,b]$ and assume $f_n \longrightarrow f$ uniformly on $(a,b)$. Prove that $f_n$ converges uniformly on $[a,b]$.*

**Proof.** We show that $f_n(x)$ is uniformly Cauchy on $[a,b]$. Let $\varepsilon > 0$ be arbitrary. As $f_n \longrightarrow f$ uniformly on $(a,b)$, there is $N \in \mathbb{N}$ such that for all $n > m > N$, and for all $x \in (a,b)$

$$|f_n(x) - f_m(x)| < \varepsilon. \tag{5}$$

This gives

$$\sup_{x \in (a,b)} |f_n(x) - f_m(x)| < \varepsilon. \tag{6}$$

As $f_n(x)$ is continuous on $[a,b]$, we have

$$\max_{x \in [a,b]} |f_n(x) - f_m(x)| = \sup_{x \in (a,b)} |f_n(x) - f_m(x)| \tag{7}$$

and the conclusion follows. □