Math 317 Winter 2017 Homework 3 Solutions

Due Thursday Mar. 9, 2017 5pm

- This homework consists of 5 problems of 4 points each. The total is 20.
- You need to fully justify your answer – for example, prove that your function indeed has the specified property – for each problem.

Question 1. (4 pts) *Prove that $\sum_{n=1}^{\infty} n^3 e^{-n}$ converges.*

Proof. We apply the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^3 e^{-(n+1)}}{n^3 e^{-n}} = \frac{1}{e}\left(\frac{n+1}{n}\right)^3. \tag{1}$$

This gives

$$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{e} < 1. \tag{2}$$

Therefore the series converges. □

Question 2. (4 pts) *Find all values of $p > 0$ such that $\sum_{n=1}^{\infty} p^n n^p$ is convergent. Justify your claim.*

Solution. We claim that the series converges if and only if $p < 1$.

- If $p \geqslant 1$, we have

$$|a_n| = p^n n^p \geqslant 1^n n^1 = n \not\longrightarrow 0. \tag{3}$$

  Thus the series diverges.
- If $0 < p < 1$, we apply the root test:

$$|a_n|^{1/n} = p\left(n^{1/n}\right)^p \longrightarrow p \in (0,1) \tag{4}$$

  as $n \longrightarrow \infty$. Consequently the series converges.

Question 3. (4 pts) *Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then for every fixed $k \in \mathbb{N}$, there holds*

$$\lim_{n\to\infty} (a_n + \cdots + a_{n+k}) = 0. \tag{5}$$

Then find a divergent series $\sum_{n=1}^{\infty} a_n$ such that for every fixed $k \in \mathbb{N}$, $\lim_{n\to\infty} (a_n + \cdots + a_{n+k}) = 0$.

Solution.

- Let $\sum_{n=1}^{\infty} a_n$ converge. Then it is Cauchy. Let $k \in \mathbb{N}$ be fixed and let $\varepsilon > 0$ be arbitrary. As $\sum_{n=1}^{\infty} a_n$ is Cauchy, there is $N \in \mathbb{N}$ such that for all $n > m > N$,

$$|a_{m+1} + \cdots + a_n| < \varepsilon. \tag{6}$$

  In particular, for every $n > N$, there holds

$$|a_n + \cdots + a_{n+k}| < \varepsilon. \tag{7}$$

  Thus by definition $\lim_{n\to\infty} (a_n + \cdots + a_{n+k}) = 0$.
- On the other hand, Let $a_n = \frac{1}{n}$. We see that

$$0 < a_n + \cdots + a_{n+k} = \frac{1}{n} + \cdots + \frac{1}{n+k} < \frac{k}{n} \longrightarrow 0. \tag{8}$$

  However we know that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Question 4. (4 pts) *Assume that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} \frac{(n+1)^2 a_n b_n}{n^2}$ converges.*

Proof. As $n \geqslant 1$, we have $(n+1)^2 \leqslant 4\,n^2$ for all n. Thus there holds

$$\left|\frac{(n+1)^2}{n^2}\,a_n\,b_n\right| \leqslant 4\,|a_n\,b_n| \leqslant 2\,(a_n^2+b_n^2). \tag{9}$$

The conclusion now follows from the comparison theorem. □

QUESTION 5. (4 PTS) *Prove or disprove: $a_n>0$, $a_n \longrightarrow 0$, then $\sum_{n=0}^{\infty} (-1)^n\,a_n$ converges.*

Solution. The claim is false. We define

$$a_n = \begin{cases} \frac{1}{k} & n = 2\,k \\ \frac{1}{k^2} & n = 2\,k+1 \end{cases}. \tag{10}$$

We show that $\sum_{n=0}^{\infty} (-1)^n\,a_n$ is not Cauchy. Take $\varepsilon_0 = \frac{1}{4}$. Let $N \in \mathbb{N}$ be arbitrary. We take k_0 such that

a) $\sum_{k \geqslant k_0} \frac{1}{k^2} < \frac{1}{4}$. Note that this is possible since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

b) $2\,k_0 > N+1$.

We set $m = 2\,k_0 - 1, n = 4\,k_0$. Then we have

$$\begin{aligned}
|a_{m+1} + \cdots + a_n| &= \left|\frac{1}{k_0} - \frac{1}{k_0^2} + \frac{1}{k_0+1} - \frac{1}{(k_0+1)^2} + \cdots + \frac{1}{2\,k_0} - \frac{1}{(2\,k_0)^2}\right| \\
&= \left|\sum_{k=k_0}^{2k_0} \frac{1}{k} - \sum_{k=k_0}^{2k_0} \frac{1}{k^2}\right| \\
&\geqslant \sum_{k=k_0}^{2k_0} \frac{1}{k} - \sum_{k=k_0}^{2k_0} \frac{1}{k^2} \\
&\geqslant \sum_{k=k_0}^{2k_0} \frac{1}{2\,k_0} - \sum_{k=k_0}^{\infty} \frac{1}{k^2} \\
&\geqslant \frac{1}{2} - \frac{1}{4} = \frac{1}{4} = \varepsilon_0.
\end{aligned} \tag{11}$$