Math 317 Winter 2017 Homework 2 Solutions

Due Thursday Feb. 16, 2017 5pm

- This homework consists of 5 problems of 4 points each. The total is 20.
- You need to fully justify your answer – for example, prove that your function indeed has the specified property – for each problem.
- This homework covers material up to and including Jan. 26 lecture.

Question 1. (4 pts) *Computer $\int_C F \cdot \mathrm{d}x$ where $F = (e^x, e^y, x+y)$ and C is the triangle joining $(1,0,0)$, $(0,1,0), (0,0,1)$ oriented in the counterclockwise direction when viewed from above.*

Solution. Let the three points be A, B, C. Then we have

$$\int_C F \cdot \mathrm{d}x = \int_{AB} F \cdot \mathrm{d}x + \int_{BC} F \cdot \mathrm{d}x + \int_{CA} F \cdot \mathrm{d}x. \tag{1}$$

- AB is parametrized by $x(t) = (1-t, t, 0)$ for $0 \leqslant t \leqslant 1$. Thus

$$\int_{AB} F \cdot \mathrm{d}x = \int_0^1 F(1-t, t, 0) \cdot (-1, 1, 0)\, \mathrm{d}t = 0. \tag{2}$$

- BC is parametrized by $x(t) = (0, 1-t, t)$ for $0 \leqslant t \leqslant 1$, and

$$\int_{BC} F \cdot \mathrm{d}x = \int_0^1 F(0, 1-t, t) \cdot (0, -1, 1)\, \mathrm{d}t = \frac{3}{2} - e. \tag{3}$$

- Similarly

$$\int_{CA} F \cdot \mathrm{d}x = -\frac{3}{2} + e. \tag{4}$$

Therefore the answer is 0.

Question 2. (4 pts) *Calculate $\int_S (z - x)\, \mathrm{d}S$ where S is the portion of the graph of $z = x + y^2$ where $0 \leqslant x \leqslant y, 0 \leqslant y \leqslant 1$.*

Solution. The parametrization is naturally $\begin{pmatrix} x \\ y \\ x+y^2 \end{pmatrix}$ with $K = \{(x, y) | \, 0 \leqslant x \leqslant y, 0 \leqslant y \leqslant 1\}$. Then we have

$$\begin{aligned}
\int_S (z-x)\, \mathrm{d}S &= \int_K [(x+y^2) - x] \left\| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} \right\| \mathrm{d}x\, \mathrm{d}y \\
&= \int_0^1 \left[\int_0^y y^2 \sqrt{2 + 4y^2}\, \mathrm{d}x \right] \mathrm{d}y \\
&= \int_0^1 y^3 \sqrt{2 + 4y^2}\, \mathrm{d}y \\
&\overset{u = 2+4y^2}{=\!=\!=} \frac{1}{32} \int_2^6 (u-2) \sqrt{u}\, \mathrm{d}u \\
&= \frac{1}{30} \left(6\sqrt{6} + \sqrt{2}\right).
\end{aligned} \tag{5}$$

Question 3. (4 pts) *Calculate $\int_S F \cdot \mathrm{d}S$ where $F = (0, 0, x)$ and S is the surface with parametrization $\Phi(u, v) = (u^2, v, u^3 - v^2)$ for $0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 1$ and oriented by upward-pointing normal vectors.*

Solution. We have

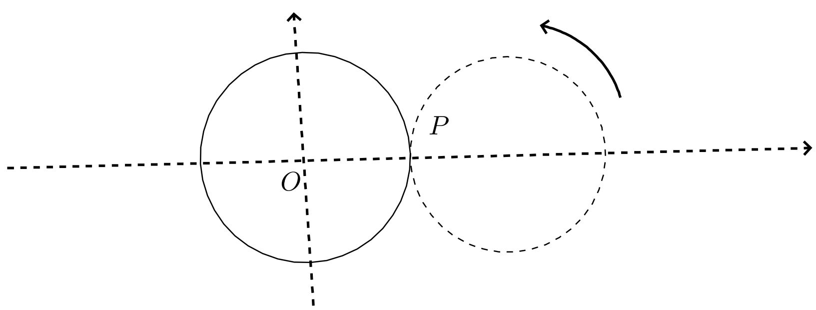
$$N(u, v) = \pm \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} = \pm \begin{pmatrix} -3u^2 \\ 4uv \\ 2u \end{pmatrix}. \tag{6}$$

As $0 \leqslant u \leqslant 1$ we have $2\,u \geqslant 0$. Therefore we should take $+$ for $N(u,v)$, that is $N(u,v) = \begin{pmatrix} -3\,u^2 \\ 4\,u\,v \\ 2\,u \end{pmatrix}$.

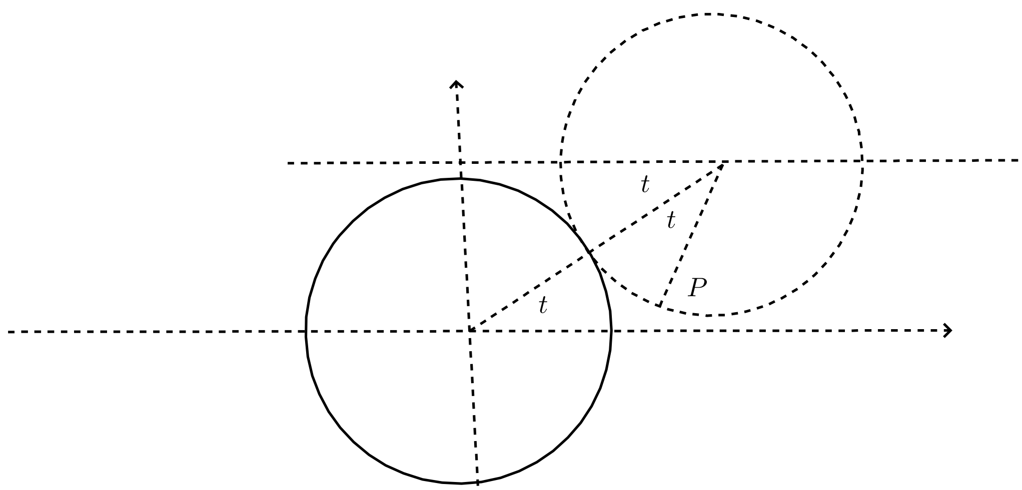
Thus

$$\int_S F \cdot \mathrm{d}S = \int_{[0,1]^2} \begin{pmatrix} 0 \\ 0 \\ u^2 \end{pmatrix} \cdot \begin{pmatrix} -3\,u^2 \\ 4\,u\,v \\ 2\,u \end{pmatrix} \mathrm{d}u\,\mathrm{d}v = \frac{1}{2}. \tag{7}$$

Question 4. (4 pts) *Write down a parametrized representation of the trajectory of a fixed point P on a unit circle rolling outside another unit circle centered at the origin. Then calculate the arc length of the curve.*



Solution. Let t be the angle as shown in the plot below.



We see that the trajectory of P is given by

$$2\,(\cos t, \sin t) - (\cos 2\,t, \sin 2\,t). \tag{8}$$

We have

$$x'(t) = 2\,(-\sin t + \sin 2\,t, \cos t - \cos 2\,t) \tag{9}$$

which leads to

$$\|x'(t)\| = 2\,\sqrt{2 - 2\cos t} \tag{10}$$

and the arc length is 16.[1]

1. The detaills are omitted here as the integral has already been calculated in the lecture note by Dr. Runde.

QUESTION 5. (4 PTS) *Let* $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$ *be two points on unit sphere centering at the origin. Prove that the shortest path on the sphere connecting them is part of a big circle.*

Solution. Let $y = (y_1, y_2, y_3), z = (z_1, z_2, z_3)$ satisfy $y_1^2 + y_2^2 + y_3^2 = z_1^2 + z_2^2 + z_3^2 = 1$. Among all curves with $x(a) = y, x(b) = z$, $x_1^2(t) + x_2^2(t) + x_3^2(t) = 1$, find the one minimizing the integral

$$L := \int_a^b \|x'(t)\| \, \mathrm{d}t. \tag{11}$$

Our goal is to show that the minimizing curve is the great arc connecting y, z. Due to the symmetry of the problem, it suffices to prove this for $y = (0, 1, 0)$ and $z = (0, 0, 1)$.

Thus all we need to show is $L \geqslant \pi/2$. For arbitrary $x(t)$ on the sphere connecting y, z, we define a new curve:

$$X(t) = (0, r(t), x_3(t)) \tag{12}$$

where $r(t) := (x_1(t)^2 + x_2(t)^2)^{1/2}$. We notice that $X(t)$ connects y, z and covers the great arc connecting y, z. Therefore the arc length of $X(t)$ is no less than $\pi/2$. For $X(t)$ we calculate

$$\begin{aligned} \frac{\pi}{2} \leqslant L_X &= \int_a^b \sqrt{r'(t)^2 + x_3'(t)^2} \, \mathrm{d}t \\ &= \int_a^b \sqrt{\frac{(x_1(t)\, x_1'(t) + x_2(t)\, x_2'(t))^2}{x_1(t)^2 + x_2(t)^2} + x_3'(t)^2} \, \mathrm{d}t \\ &\leqslant \int_a^b \sqrt{x_1'(t)^2 + x_2'(t)^2 + x_3'(t)^2} \, \mathrm{d}t = L. \end{aligned} \tag{13}$$