Math 317 Winter 2017 Homework 1

Due Thursday Feb. 2, 2017 5pm

- This homework consists of 5 problems of 4 points each. The total is 20.
- You need to fully justify your answer – for example, prove that your function indeed has the specified property – for each problem.
- This homework covers material up to and including Jan. 26 lecture.

Question 1. (4 pts) *Consider the function $T: \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined through*

$$T(x, y) = \begin{pmatrix} x^2 - y^2 \\ 2\,x\,y \end{pmatrix}. \tag{1}$$

Show that T does not have an inverse function on any open set $U \ni (0,0)$.

Solution. Let U be an open set containing $(0,0)$. Then there is $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(0) \subseteq U$. In particular there are $(x_0, y_0) \in U$ such that $(-x_0, -y_0) \in U$ too. As $T(x_0, y_0) = T(-x_0, -y_0)$ there can be no inverse function on U.

Question 2. (4 pts) *Consider the function $T: \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined through*

$$T(x, y) = \begin{pmatrix} x^2 - y^2 \\ 2\,x\,y \end{pmatrix}. \tag{2}$$

We accept that T has an inverse function near $x = 1$, $y = 1$. Calculate the Jacobian matrix of the inverse function at the corresponding point.

Solution. We have

$$T(1,1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{3}$$

Denote by G the inverse function of T around $(1,1)$. Then G is defined around $(0,2)$, and furthermore

$$J_G(0,2) = J_T(1,1)^{-1} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{4}$$

Question 3. (4 pts) *Let an implicit function $Y(x)$ be defined through*

$$y^3 + x^2\,y^2 - x\,y + x^4 = 0 \tag{5}$$

Calculate $Y'(0), Y''(0), Y'''(0)$.

Solution. Replacing y by Y we have

$$Y^3 + x^2\,Y^2 - x\,Y + x^4 = 0. \tag{6}$$

Taking derivative

$$3\,Y^2\,Y' + 2\,x\,Y^2 + 2\,x^2\,Y Y' - Y - x\,Y' + 4\,x^3 = 0. \tag{7}$$

Setting $x = 0$ and using $Y(0) = 0$ we obtain $0 = 0$.

Taking derivative again, we obtain

$$3\,Y^2\,Y'' + 6\,Y Y'^2 + 2\,Y^2 + 8\,x\,Y Y' + 2\,x^2\,Y'^2 + 2\,x\,Y Y'' - 2\,Y' - x\,Y'' + 12\,x^2 = 0. \tag{8}$$

Setting $x = 0$ we have $Y' = 0$.

Now we notice the following. Each of the brown terms involve a product of Y terms. If we differentiate them, we will have a sum of products of Y and its derivatives, with at least one factor either Y or Y'. Therefore, if we differentiate one more time and set $x = 0$, we reach

$$-3\,Y'' = 0 \Longrightarrow Y'' = 0. \tag{9}$$

By similar argument, one more differentiation gives

$$-4\,Y''' + 24 = 0 \Longrightarrow Y'''(0) = 6. \tag{10}$$

QUESTION 4. *Let $f\colon \mathbb{R}^3 \mapsto \mathbb{R}$ be a C^1 function. Consider the equation $f(x,y,z)=0$. Assume that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are all nonzero at (x_0,y_0,z_0) and thus around (x_0,y_0,z_0) we can define x as an implicit function of y,z, y as an implicit function of x,z, and z as an implicit function of x,y. Prove that $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$.*

Solution. Let $x = X(y,z)$. Differentiating $f(X(y,z),y,z)=0$ we have

$$\frac{\partial X}{\partial y} = -\left(\frac{\partial f}{\partial x}\right)^{-1}\frac{\partial f}{\partial y}. \tag{11}$$

Similarly we have

$$\frac{\partial Y}{\partial z} = -\left(\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial f}{\partial z}, \qquad \frac{\partial Z}{\partial x} = -\left(\frac{\partial f}{\partial z}\right)^{-1}\frac{\partial f}{\partial x}. \tag{12}$$

Clearly we have $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$.

QUESTION 5. *Find the maximum value of $\left|\sum_{k=1}^N a_k\,x_k\right|$ subject to $\sum_{k=1}^N x_k^2 = 1$ using the Lagrange multiplier method.*

Solution. Denote

$$R^2 := \sum_{k=1}^N a_k^2. \tag{13}$$

Let

$$f(x_1,\ldots,x_N) := \left(\sum_{k=1}^N a_k\,x_k\right)^2. \tag{14}$$

We define the Lagrange function

$$L(x_1,\ldots,x_N,\lambda) := \left(\sum_{k=1}^N a_k\,x_k\right)^2 - \lambda\left(\sum_{k=1}^N x_k^2 - 1\right). \tag{15}$$

The equations for stationary points are then

$$2\,a_1\sum_{k=1}^N a_k\,x_k - 2\,\lambda\,x_1 = \frac{\partial L}{\partial x_1} = 0, \tag{16}$$

$$2\,a_2\sum_{k=1}^N a_k\,x_k - 2\,\lambda\,x_2 = \frac{\partial L}{\partial x_2} = 0, \tag{17}$$

$$\vdots$$

$$2\,a_N\sum_{k=1}^N a_k\,x_k - 2\,\lambda\,x_N = \frac{\partial L}{\partial x_N} = 0. \tag{18}$$

We see that there are two cases:

i. $\sum_{k=1}^N a_k\,x_k = 0$. In this case we see that $f(x_1,\ldots,x_N)=0$ so this corresponds to a minimizer.

ii. $\sum_{k=1}^N a_k\,x_k \neq 0$. In this case we have

$$\frac{a_1}{x_1} = \frac{a_2}{x_2} = \cdots = \frac{a_N}{x_N} = \frac{\lambda}{\sum_{k=1}^N a_k\,x_k} \tag{19}$$

which gives $f(x_1,\ldots,x_N) = R^2$.

As $\sum_{k=1}^N x_k^2 = 1$ is a smooth compact set, f attains its minimum and maximum at stationary points. Consequently the maximum of f is $\sum_{k=1}^N a_k^2$ and the maximum of $\left|\sum_{k=1}^N a_k\,x_k\right|$ is $\sqrt{\sum_{k=1}^N a_k^2}$.