Problem 1. Determine the convergence set of the given power series.

a) \( \sum_{n=1}^{\infty} \frac{3}{n^8} (x-2)^n \),

b) \( \sum_{n=0}^{\infty} 2^n x^{3n} \).

Solution.

a) For this power series we have \( a_0 = 0, a_n = \frac{3}{n^8} \) for \( n \geq 1 \). As the ratio test is about \( n \to \infty \), \( a_0 \) doesn’t matter. We have

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^8}{n^8} \right| = \left| \frac{n+1}{n} \right|^8 \to 1 \quad \text{as} \quad n \to \infty.
\]

So the radius of convergence is \( \rho = 1^{-1} = 1 \). The power series converges for \( |x-2|<1 \) and diverges for \( |x-2|>1 \).

Now check \( |x-2|=1 \) that is \( x=1, 3 \). In this case we have

\[
\left| \frac{3}{n^8} (x-2)^n \right| = \frac{3}{n^8} |x-2|^n = \frac{3}{n^8}.
\]

As

\[
\sum_{n=1}^{\infty} \frac{3}{n^8}
\]

converges, the original power series also converges at \( x=1, 3 \). (Here we have used the following property of infinite sum: If \( |a_n| \leq b_n \) and \( \sum b_n \) converges, then \( \sum a_n \) converges too.)

Summarizing, the convergence set is \( |x-2| \leq 1 \) or equivalently \( 1 \leq x \leq 3 \).

b) For this one we first set \( t = x^3 \) and consider

\[
\sum 2^n t^n.
\]

We have \( a_n = 2^n \) so

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{2^{(n+1)}}{2^n} \right| = 2 \to 2 \quad \text{as} \quad n \to \infty.
\]

Thus the power series in \( t \) has radius of convergence \( 1/2 \), that is converges for \( |t| < 1/2 \) and diverges when \( |t| > 1/2 \). When \( t = \pm 1/2 \) we have

\[
\sum 2^n t^n |t=1/2 \implies \sum_{n=0}^{\infty} 1; \quad \sum 2^n t^n |t=-1/2 \implies \sum (-1)^n
\]

both clearly diverge. So the convergence set for \( t \) is \( |t| < 1/2 \).

As \( t = x^3 \), the convergence set for \( x \) is

\[
|x| < \left( \frac{1}{2} \right)^{1/3}.
\]

Problem 2. Let \( f(x) \) and \( g(x) \) be defined by the following two power series:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n, \quad \sum_{n=0}^{\infty} 2^n x^{n+1}.
\]

a) For what \( x \) are \( f \) and \( g \) both defined? (Only for these \( x \)'s is \( f \pm g, fg, \) etc meaningful).

b) Obtain the power series expansion of \( 3f-2g \). Your answer should have generic term \( x^n \).

c) Find the first three nonzero terms in the power series expansion of \( fg \).

d) Find the first three terms in the ratio \( f/g \).
e) Find the power series expansion for \( f' \) and \( \int_{0}^{x} g \). Your answer should have generic term \( x^n \).

**Solution.**

a) We easily obtain that the radii of convergence are 1 and \( \frac{1}{2} \). We can also check that the first power series converges for \( x = 1 \) but diverges for \( x = -1 \), while the second power series diverges for both \( x = \pm \frac{1}{2} \). So \( f \) is defined for \(-1 < x \leq 1\) while \( g \) for \( |x| < 1/2 \). Therefore both are defined over \( |x| < 1/2 \).

b) We have

\[
3 f - 2 g \sim 3 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n - 2 \sum_{n=0}^{\infty} 2^n x^{n+1}
\]

\[
= \sum_{n=1}^{\infty} \frac{3 (-1)^n}{n} x^n - \sum_{n=0}^{\infty} 2^{n+1} x^{n+1}
\]

(9)

Now we shift index in the 2nd sum: (Setting \( m = n + 1 \))

\[
\sum_{n=0}^{\infty} 2^{n+1} x^{n+1} = \sum_{m=1}^{\infty} 2^m x^m = \sum_{m=1}^{\infty} 2^m x^m.
\]

(10)

Now rename \( m \) to \( n \):

\[
\sum_{m=1}^{\infty} 2^m x^m = \sum_{n=1}^{\infty} 2^n x^n.
\]

(11)

Thus

\[
3 f - 2 g \sim \sum_{n=1}^{\infty} \frac{3 (-1)^n}{n} x^n - \sum_{n=1}^{\infty} 2^n x^n = \sum_{n=1}^{\infty} \left[ \frac{3 (-1)^n}{n} - 2^n \right] x^n.
\]

(12)

c) To do this the best way is to first expand \( f \) and \( g \): (Since the first three terms are required, we expand to 3 terms first and see if that’s enough)

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n = -x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots; \quad \sum_{n=0}^{\infty} 2^n x^{n+1} = x + 2x^2 + 4x^3 + \cdots
\]

(13)

Now take the product:

\[
f g \sim \left( -x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots \right) \left( x + 2x^2 + 4x^3 + \cdots \right)
\]

\[
= -x^2 + \left[ \frac{x^2}{2} \cdot x + (-x) \cdot 2x^2 \right] + \left[ \left( -\frac{x^3}{3} \right) x + \frac{x^2}{2} \cdot 2x^2 + (-1)4x^3 \right] + \text{higher order terms}
\]

\[
= -x^2 + \frac{3}{2} x^3 - \frac{10}{3} x^4 + \cdots.
\]

We already have three nonzero terms, so done.

d) Again we expand the power series to first three terms:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n = -x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots; \quad \sum_{n=0}^{\infty} 2^n x^{n+1} = x + 2x^2 + 4x^3 + \cdots
\]

(14)

Long division then gives (see 8.2 16):

\[
\frac{f}{g} \sim -1 + \frac{5}{2} x - \frac{4}{3} x^2 + \cdots
\]

(15)

Note that both power series are 0 at \( x = 0 \) due to

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n = \left( -1 + \frac{x}{2} - \frac{x^2}{3} + \cdots \right); \quad \sum_{n=0}^{\infty} 2^n x^{n+1} = x \left( 1 + 2x + 4x^2 + \cdots \right)
\]

(16)
However the two $x$'s cancel in $f/g$. So $f/g$ is still well-defined.

e) We have
\[ f' \sim \sum_{n=1}^{\infty} \left[ \frac{(-1)^n}{n} x^n \right]' = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} n x^{n-1} = \sum_{n=1}^{\infty} (-1)^n x^{n-1}. \] (17)

Now we shift index. Let $m = n - 1$ we have
\[ \sum_{n=1}^{\infty} (-1)^n x^{n-1} = \sum_{m=1}^{\infty} (-1)^{m+1} x^m = \sum_{m=0}^{\infty} (-1)^{m+1} x^m. \] (18)

Renaming $m$ to $n$ we reach
\[ f' \sim \sum_{n=0}^{\infty} (-1)^{n+1} x^n. \] (19)

For $g$ we have
\[ \int_0^x g(t) \, dt \sim \int_0^x \sum_{n=0}^{\infty} 2^n t^{n+1} \, dt = \sum_{n=0}^{\infty} 2^n \int_0^x t^{n+1} \, dt = \sum_{n=0}^{\infty} 2^n \frac{x^{n+2}}{n+2}. \] (20)

Shifting index $n + 2 \rightarrow n$ we finally obtain
\[ \int_0^x g(t) \, dt \sim \sum_{n=2}^{\infty} \frac{2n-2}{n} x^n. \] (21)

**Problem 3. (8.2 36)** Let $f(x)$ and $g(x)$ be analytic at $x_0$. Determine whether the following statements are always true or sometimes false.

a) $3f(x) + g(x)$ is analytic at $x_0$;

b) $f(x)/g(x)$ is analytic at $x_0$;

c) $f'(x)$ is analytic at $x_0$;

d) $(f(x))^3 - f_{x_0}^x g(t) \, dt$ is analytic at $x_0$.

**Solution.**

a) True; b) False (because $g(x_0)$ may be 0); c) True; d) True.

**Problem 4.** Is
\[ \frac{e^x - \sin x - \cos x^2}{x^2} \] (22)

analytic at 0 or not?

**Solution.** We know that $e^x$, $\sin x$, $\cos x^2$, $x^2$ are all analytic for all $x$. However $x^2 = 0$ at $x = 0$ so it is still possible that the ratio is not analytic at $x = 0$. On the other hand we easily check that $e^x - \sin x - \cos x^2 = 0$ at $x = 0$ too. So it is also possible that the ratio is analytic at $x = 0$.

To get a definite answer we expand the numerator: For all $x$ the following holds true:
\[ e^x - \sin x - \cos x^2 = \left[ 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots \right] - \left[ x - \frac{x^3}{6} + \ldots \right] - \left[ 1 - \frac{(x^2)^2}{2} + \ldots \right] = x^2 \left[ \frac{1}{2} + \frac{x}{3} + \frac{13}{24} x^2 + \ldots \right] \] (23)

which leads to
\[ \frac{e^x - \sin x - \cos x^2}{x^2} = \frac{1}{2} + \frac{x}{3} + \frac{13}{24} x^2 + \ldots \] (24)

so $f/g$ is analytic at $x = 0$.

**Problem 5.** Determine all the singular points for the given differential equations. Then determine the lower bound of the radius of convergence for the power series solutions at $x_0 = 2$.

a) $(1 + x^3) y'' - xy'y' + 3 x^2 y = 0;$
b) \( \sin(x)\, y'' + e^x\, y = 0; \)

c) \( \sin(x)\, y'' - (\ln x)\, y = 0. \)

d) \( \sin(x)\, y'' + y = 0. \)

**Solution.**

a) Write the equation into standard form:

\[
y'' - \frac{x}{(1 + x^3)} \, y' + \frac{3\, x^2}{1 + x^3} \, y = 0. \quad (25)
\]

So \( p(x) = -\frac{x}{(1 + x^3)}, \quad q(x) = \frac{3\, x^2}{1 + x^3}. \) As both are ratios of polynomials, which are analytic everywhere, we only need to check the zeroes of \( 1 + x^3. \)

To solve \( 1 + x^3 = 0, \) first observe that \( x_1 = -1 \) is a solution. Now factorize

\[
1 + x^3 = (x + 1)\, (x^2 - x + 1) \implies x_{2,3} = \frac{1 \pm \sqrt{3} \, i}{2}. \quad (26)
\]

As \( x \) and \( 3\, x^2 \) do not vanish at \( x_1, x_2, x_3, \) \( p(x), q(x) \) are not analytic at \( x_1, x_2, x_3. \) Thus the singular points are

\[
-1, \frac{1 \pm \sqrt{3} \, i}{2}. \quad (27)
\]

To find the lower bound of the radius of convergence at \( x_0 = 2, \) we calculate the distances:

\[
|2 - (-1)| = 3; \quad \left| 2 - \left( \frac{1 \pm \sqrt{3} \, i}{2} \right) \right| = \frac{3}{2} \mp \frac{\sqrt{3}}{2} \, i = \sqrt{3}. \quad (28)
\]

The shortest distance is \( \sqrt{3}. \) So the lower bound for the radius of convergence is \( \sqrt{3}. \)

b) Write to standard form

\[
y'' + \frac{e^x}{\sin x} \, y = 0. \quad (29)
\]

Both \( e^x \) and \( \sin x \) are analytic everywhere so we check zeroes of \( \sin x \) which is \( n\, \pi \) for integers \( n. \)

As \( e^x \neq 0 \) at \( x = n\, \pi \) for any \( n, \) the singular points are \( \{ n\, \pi \} \) for integers \( n. \)

It is easy to see that the smallest \( |2 - n\, \pi| \) happens when \( n = 1. \) So the lower bound is \( |2 - \pi| = \pi - 2. \)

c) The equation is already in standard form \( y'' - (\ln x)\, y = 0. \) The singular points are \( x \leq 0. \) The shortest distance of \( x_0 = 2 \) to this singular point set is clearly \( 2. \) So the lower bound is \( 2. \)

d) The equation \( y'' - (\tan x)\, y' + y = 0 \) is already in standard form. Its singular points are those points where \( -\tan x = -\frac{\sin x}{\cos x} \) is not analytic. We find out that these points are those such that \( \cos x = 0: \quad x = (n + 1/2)\, \pi. \) The shortest distance of \( 2 \) to these points is reached at \( n = 0. \) Thus the lower bound is \( 2 - \pi/2. \)

**Problem 6. (8.3 14)** Find at least the first four nonzero terms in a power series expansion about \( x = 0 \) for a general solution to the given differential equation.

\[
(x^2 + 1)\, y'' + y = 0. \quad (30)
\]

**Solution.** Since we only need first four terms, we write \( y \) up to four terms (Note that if one of \( a_0 \cdots a_3 \) turns out to be 0, we need to return here and expand more):

\[
y = a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 \cdots \quad (31)
\]

and substitute into the equation.

\[
(x^2 + 1)\, [2\, a_2 + 6\, a_3\, x \cdots] + [a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \cdots] = 0. \quad (32)
\]
Expanding the left hand side we reach:
\[0 = x^2 [2a_2 + 6a_3 x + \ldots] + [2a_2 + 6a_3 x + \ldots] + [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots] = (2a_2 + a_0) + (6a_3 + a_1) x + \ldots\]  \hspace{1cm} (33)

Thus
\[2a_2 + a_0 = 0, \quad 6a_3 + a_1 = 0\]  \hspace{1cm} (34)

and we have \(y\) up to first four nonzero terms:
\[y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = a_0 + a_1 x - a_0 \frac{x^2}{2} - a_1 \frac{x^3}{6} + \ldots\]  \hspace{1cm} (35)

**Problem 7. (8.3 21)** Solve
\[y'' - x y' + 4 y = 0.\]  \hspace{1cm} (36)

**Solution.** Since we are asked to “solve”, we need to write
\[y = \sum_{n=0}^{\infty} a_n x^n\]  \hspace{1cm} (37)

and substitute into the equation. We have
\[y'' = \sum_{n=2}^{\infty} n (n - 1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n + 2) (n + 1) a_{n+2} x^n.\]  \hspace{1cm} (38)

\[x y' = x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^n\]  \hspace{1cm} (39)

after index shifting. The equation now reads
\[\sum_{n=0}^{\infty} (n + 2) (n + 1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0.\]  \hspace{1cm} (40)

As the sums start from different values, we have to break the two \(\sum_{n=0}^{\infty}\) into \(n = 0\) term + \(\sum_{n=1}^{\infty}\) :
\[\sum_{n=0}^{\infty} (n + 2) (n + 1) a_{n+2} x^n = 2a_2 + \sum_{n=1}^{\infty} (n + 2) (n + 1) a_{n+2} x^n;\]  \hspace{1cm} (41)

\[\sum_{n=0}^{\infty} 4 a_n x^n = 4a_0 + \sum_{n=1}^{\infty} 4 a_n x^n.\]  \hspace{1cm} (42)

Now the equation can be written as
\[2a_2 + 4a_0 + \sum_{n=1}^{\infty} [(n + 2) (n + 1) a_{n+2} - (n - 4) a_n] x^n = 0\]  \hspace{1cm} (43)

which gives the following recurrence relations:
\[2a_2 + 4a_0 = 0;\]  \hspace{1cm} (44)

\[(n + 2) (n + 1) a_{n+2} - (n - 4) a_n = 0 \implies a_{n+2} = \frac{n - 4}{(n + 2) (n + 1)} a_n\]  \hspace{1cm} (45)

Clearly \(a_0\) determines \(a_2\), then \(a_4\), then \(a_6\), ... and \(a_1\) determines \(a_3\), \(a_5\), \(a_7\), ...

Looking more closely, we realize that setting \(n = 4\) in the general recurrence relation gives
\[a_6 = \frac{4}{(4 + 2) (4 + 1)} a_4 = 0.\]  \hspace{1cm} (46)

And then
\[a_8 = \frac{6 - 4}{(6 + 2) (6 + 1)} a_6 = 0.\]  \hspace{1cm} (47)
In general, we have \( a_{2k} = 0 \) for all \( k \geq 3 \). For \( k < 3 \), we have
\[
a_2 = -2a_0, \quad a_4 = \frac{2 - 4}{(2 + 2) (2 + 1)} = -\frac{a_2}{6} = \frac{a_0}{3}.
\]

On the other hand, for odd \( n \), setting \( n = 2k + 1 \), we have
\[
a_{2k+1} = \frac{2k - 5}{(2k+1)(2k)} a_{2k-1} = \frac{(2k - 5)(2k - 7)}{(2k+1)(2k)(2k - 1)(2k - 2)} a_{2k-2} = \ldots = \frac{(2k - 5) \cdots (-1)(-3)}{(2k + 1)!} a_1.
\]

So the solution is
\[
y = a_0 \left[ 1 - 2x^2 + \frac{x^4}{3} \right] + a_1 \left[ x + \sum_{k=1}^{\infty} \frac{(2k - 5) \cdots (-1)(-3)}{(2k + 1)!} x^{2k+1} \right].
\]

**Problem 8. (8.3 22)** Solve the Airy’s equation:
\[
y'' - xy = 0.
\]

**Solution.** Write
\[
y = \sum_{n=0}^{\infty} a_n x^n.
\]

Then
\[
y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.
\]

after index shifting.

On the other hand
\[
x y = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n
\]

after index shifting.

The equation becomes
\[
\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.
\]

Since the two sums start from different values, we have to break the first sum into \( n = 0 \) term + \( \sum_{n=1}^{\infty} \):
\[
\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n.
\]

Now the equation becomes
\[
2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-1}] x^n = 0
\]

which leads to the recurrence relations:
\[
2a_2 = 0; \quad (n+2)(n+1) a_{n+2} - a_{n-1} = 0 \implies a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}.
\]

We see that \( a_0 \) determines \( a_3, a_6, a_9, \ldots \), \( a_1 \) determines \( a_4, a_7, a_{10}, \ldots \), and \( a_2 \) determines \( a_5, a_8, a_{11}, \ldots \).

More specifically, we have
\[
a_{3k} = \frac{a_{3k-3}}{(3k)(3k-1)} = \frac{a_{3k-6}}{(3k)(3k-1)(3k-3)} = \ldots = \frac{a_0}{(3k)(3k-1) \cdots 5 \cdot 3 \cdot 2};
\]
\[
a_{3k+1} = \frac{a_{1}}{(3k+1)(3k) \cdots 7 \cdot 6 \cdot 4 \cdot 3};
\]

\[
a_{3k+2} = \frac{a_{3k}}{(3k+2)(3k+1)(3k-1) \cdots 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}.
\]
and
\[ a_{3k+2} = \frac{a_2}{(3k+2)(3k+1)\ldots8\cdot7\cdot5\cdot4} = 0. \]  
(62)

So the solution is
\[ y = a_0 \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)(3k-1)\ldots6\cdot3\cdot2} + a_1 \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)(3k)\ldots7\cdot6\cdot4\cdot3}. \]  
(63)

**Problem 9. (8.3 24)** Find a power series expansion about \( x = 0 \) for a general solution to the given differential equation. Your answer should include a general formula for the coefficients.

\[ (x^2 + 1) y'' - x y' + y = 0 \]  
(64)

**Solution.** As we need “general formula for the coefficients”, we write
\[ y = \sum_{n=0}^{\infty} a_n x^n \]  
(65)

and substitute into the equation. We calculate
\[ (x^2 + 1) y'' - x y' + y = x^2 \left( \sum_{n=0}^{\infty} a_n x^n \right)'' + \left( \sum_{n=0}^{\infty} a_n x^n \right)' - x \left( \sum_{n=0}^{\infty} a_n x^n \right) + \sum_{n=0}^{\infty} a_n x^n \]
\[ = x^2 \sum_{n=2}^{\infty} a_n n (n-1) x^{n-2} + \sum_{n=2}^{\infty} a_n n (n-1) x^{n-2} - x \sum_{n=1}^{\infty} a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \]
\[ = \sum_{n=2}^{\infty} a_n n (n-1) x^n + \sum_{n=0}^{\infty} a_{n+2} (n+2) (n+1) x^n - \sum_{n=1}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^n \]
\[ = \sum_{n=2}^{\infty} a_n n (n-1) x^n + 2 a_2 + 6 a_3 x + \sum_{n=2}^{\infty} a_{n+2} (n+2) (n+1) x^n - a_1 x - \sum_{n=2}^{\infty} a_n n x^n + a_0 + a_1 x + \sum_{n=0}^{\infty} a_n x^n \]
\[ = (2 a_2 + a_0) + (6 a_3) x + \sum_{n=2}^{\infty} [a_n n (n-1) - n a_n + a_n + a_{n+2} (n+2) (n+1)] x^n \]
\[ = (2 a_2 + a_0) + (6 a_3) x + \sum_{n=2}^{\infty} [a_{n+2} (n+2) (n+1) + a_n (n-1)^2] x^n. \]
So we have the following:

\[ 2a_2 + a_0 = 0 \]  \hspace{1cm} (66)
\[ 6a_3 = 0 \]  \hspace{1cm} (67)
\[ a_{n+2}(n+2)(n+1)+a_n(n-1)^2 = 0 \text{ for all } n \geq 2 \]  \hspace{1cm} (68)

which gives

\[ a_2 = -\frac{a_0}{2}, \quad a_3 = 0, \quad a_{n+2} = -\frac{(n-1)^2}{(n+2)(n+1)}a_n \text{ for } n \geq 2. \]  \hspace{1cm} (69)

Note that from the 2nd and the 3rd relation we conclude that \( a_n = 0 \) for all odd \( n \geq 3 \). To derive a general formula for even \( n \), denote \( n = 2k \). We first rewrite

\[ a_{n+2} = -\frac{(n-1)^2}{(n+2)(n+1)}a_n \]  \hspace{1cm} (70)

to

\[ a_n = -\frac{(n-3)^2}{n(n-1)}a_{n-2}. \]  \hspace{1cm} (71)

From this we obtain

\[ a_{2k} = -\frac{(2k-3)^2}{2k(2k-1)}a_{2(k-1)} \]
\[ = (-1)^2 \frac{(2k-3)^2(2k-1)^2}{(2k)(2k-1)(2k-1)(2k-1)}a_{2(k-2)} \]
\[ = (-1)^2 \frac{(2k-3)(2k-5)^2}{(2k)(2k-3)}a_{2(k-2)} \]
\[ = (-1)^3 \frac{(2k-3)(2k-5)^2}{(2k)(2k-3)(2k-5)}a_{2(k-3)} \]
\[ \vdots \]
\[ = (-1)^{k-1} \frac{(2k-3)(2k-5)\cdots 1}{(2k)\cdots (3)}a_2 \]
\[ = (-1)^k \frac{(2k-3)(2k-5)\cdots 1(-1)^2}{(2k)!}a_0. \]  \hspace{1cm} (72)

Summarizing, we have

\[ y = a_1 x + a_0 \left[ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-3)(2k-5)\cdots 1(-1)^2}{(2k)!} x^{2k} \right]. \]  \hspace{1cm} (73)

Note that since \( a_0, a_1 \) are free, the two linearly independent solutions to the equation can be chosen as

\[ y_1 = x, \quad y_2 = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-3)(2k-5)\cdots 1(-1)^2}{(2k)!} x^{2k}. \]  \hspace{1cm} (74)

Although it’s not possible to check whether \( y_2 \) is really a solution or not (besides essentially repeating the above calculation), it is easy to see that \( y_1 = x \) indeed solves the equation. This improves the likelihood that our calculation is correct.

**Problem 10. (8.3 26)** Find at least first four nonzero terms in a power series expansion about \( x = 0 \) for the solution to the given initial value problem.

\[ (x^2 - x + 1) y'' - y' - y = 0; \quad y(0) = 0, y'(0) = 1. \]  \hspace{1cm} (75)

**Solution.** Write

\[ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]  \hspace{1cm} (76)

Using the initial condition we get

\[ a_0 = 0, a_1 = 1. \]  \hspace{1cm} (77)
Simplify and collect terms of same power together:

\[ y = x + a_2 x^2 + a_3 x^3 + \ldots \]  \hspace{1cm} (78)

We see that even if \( a_2, a_3 \neq 0 \) we still do not have four nonzero terms. So we need to expand at least one more term:

\[ y = x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \]  \hspace{1cm} (79)

Substitute into the equation we obtain

\[(x^2 - x + 1) \left( 2 a_2 + 6 a_3 x + 12 a_4 x^2 + \ldots \right) - (1 + 2 a_2 x + 3 a_3 x^2 + 4 a_4 x^3 + \ldots) - (x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots) = 0. \]  \hspace{1cm} (80)

Simplify the left hand side:

\[ (2 a_2 - 1) + (6 a_3 - 4 a_2 - 1) x + (a_2 - 9 a_3 + 12 a_4) x^2 + \cdots = 0 \]  \hspace{1cm} (81)

Therefore

\[ 2 a_2 - 1 = 0, \quad 6 a_3 - 4 a_2 - 1 = 0, \quad a_2 - 9 a_3 + 12 a_4 = 0, \]  \hspace{1cm} (82)

which gives

\[ a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{2}, \quad a_4 = \frac{1}{3}. \]  \hspace{1cm} (83)

So we have the first four terms:

\[ y = x + \frac{1}{2} x^2 + \frac{1}{2} x^3 + \frac{1}{3} x^4 + \ldots \]  \hspace{1cm} (84)

**Problem 11.** (8.4 12) Find at least the first four nonzero terms in a power series expansion about \( x_0 \) for a general solution to the given differential equation with the given value for \( x_0 \).

\[ y'' + (3 x - 1) y' - y = 0; \quad x_0 = -1. \]  \hspace{1cm} (85)

**Solution.** As \( x_0 = -1 \neq 0 \), we need to do a change of variable \( t = x - x_0 = x + 1 \) first. Under this change of variable the equation becomes

\[ y'' + (3 (t - 1) - 1) y' - y = 0, \quad t_0 = 0 \]  \hspace{1cm} (86)

which simplifies to

\[ y'' + (3 t - 4) y' - y = 0. \]  \hspace{1cm} (87)

Write \( y \) up to first four terms:

\[ y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots \]  \hspace{1cm} (88)

and substitute into the equation:

\[ [2 a_2 + 6 a_3 t + \ldots] + (3 t - 4) [a_1 + 2 a_2 t + 3 a_3 t^2 + \ldots] - [a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots] = 0. \]  \hspace{1cm} (89)

Simplify and collect terms of same power together:

\[ (2 a_2 - 4 a_1 - a_0) + (6 a_3 + 2 a_1 - 8 a_2) t + \cdots = 0 \]  \hspace{1cm} (90)

This gives

\[ 2 a_2 - 4 a_1 - a_0 = 0; \quad 6 a_3 + 2 a_1 - 8 a_2 = 0 \]  \hspace{1cm} (91)

so

\[ a_2 = 2 a_1 + \frac{a_0}{2}; \quad a_3 = \frac{7}{3} a_1 + \frac{2}{3} a_0. \]  \hspace{1cm} (92)

We already have \( y \) up to four nonzero terms:

\[ y = a_0 + a_1 t + \left( 2 a_1 + \frac{a_0}{2} \right) t^2 + \left( \frac{7}{3} a_1 + \frac{2}{3} a_0 \right) t^3 + \ldots \]  \hspace{1cm} (93)

Finally, back to \( x \):

\[ y = a_0 + a_1 (x + 1) + \left( 2 a_1 + \frac{a_0}{2} \right) (x + 1)^2 + \left( \frac{7}{3} a_1 + \frac{2}{3} a_0 \right) (x + 1)^3 + \ldots \]  \hspace{1cm} (94)
Problem 12. (8.4 18) Find at least the first four nonzero terms in a power series expansion of the solution to the given initial value problem.

\[ y'' - (\cos x) y' - y = 0, \quad y(\pi/2) = 1, \quad y'(\pi/2) = 0. \]  

Solution. As \( x_0 = \frac{\pi}{2} \neq 0 \), we need to change variable: \( t = x - \frac{\pi}{2} \). So that the initial conditions for \( y(t) \) becomes

\[ y(0) = 1, \quad y'(0) = 0. \]

Under this change of variable, \( \cos x = \cos \left( t + \frac{\pi}{2} \right) = -\sin t \). So the equation in \( t \) is

\[ y'' + (\sin t) y' - y = 0, \quad y(0) = 1, \quad y'(0) = 0. \]

Writing

\[ y = a_0 + a_1 t + \ldots \]

the initial conditions give

\[ a_0 = 1, \quad a_1 = 0. \]

So to obtain first four nonzero terms we have to at least write

\[ y = 1 + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots \]

and substitute into the equation.

We get

\[ [2 a_2 + 6 a_3 t + 12 a_4 t^2 + \ldots] + (t - \frac{t^3}{6} + \ldots)[2 a_2 t + 3 a_3 t^2 + 4 a_4 t^3 + \ldots] \]

\[ -[1 + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots] = 0. \]

Expanding and balancing, we get

\[ (2 a_2 - 1) + 6 a_3 t + (12 a_4 + a_2) t^2 + \ldots = 0 \]

which gives

\[ a_2 = \frac{1}{2}, \quad a_3 = 0, \quad a_4 = -\frac{1}{24}. \]

So we obtain

\[ y = 1 + \frac{1}{2} t^2 - \frac{1}{24} t^4 + \ldots \]

and do not have enough nonzero terms.

So we have to expand \( y \) up to at least one more term:

\[ y = 1 + \frac{1}{2} t^2 - \frac{1}{24} t^4 + a_5 t^5 + \ldots \]

and substitute into the equation:

\[ \left[ 1 - \frac{1}{2} t^2 + 20 a_5 t^3 + \ldots \right] + \left( t - \frac{t^3}{6} + \ldots \right) \left[ t - \frac{t^3}{6} + 5 a_5 t^4 + \ldots \right] \]

\[ -\left[ 1 + \frac{1}{2} t^2 - \frac{1}{24} t^4 + a_5 t^5 + \ldots \right] = 0. \]

\( a_5 \) is then determined by balancing the \( t^3 \) terms:

\[ 20 a_5 = 0 \Rightarrow a_5 = 0. \]

This means we need to expand even one more term:

\[ y = 1 + \frac{1}{2} t^2 - \frac{1}{24} t^4 + a_6 t^6 + \ldots \]

Substituting into the equation we get

\[ \left[ 1 - \frac{1}{2} t^2 + 30 a_6 t^4 + \ldots \right] + \left( t - \frac{t^3}{6} + \ldots \right) \left[ t - \frac{t^3}{6} + 6 a_6 t^5 + \ldots \right] \]

\[ -\left[ 1 + \frac{1}{2} t^2 - \frac{1}{24} t^4 + a_6 t^6 + \ldots \right] = 0 \]
Balancing \( t^4 \) we get

\[
30 a_6 - \frac{1}{3} + \frac{1}{24} = 0 \implies a_6 = \frac{7}{720}.
\] (109)

So finally we have obtained \( y \) up to four nonzero terms:

\[
y = 1 + \frac{1}{2} t^2 - \frac{1}{24} t^4 + \frac{7}{720} t^6 + \ldots
\] (110)

Back to \( x \):

\[
y = 1 + \frac{1}{2} \left( x + \frac{\pi}{2} \right)^2 - \frac{1}{24} \left( x + \frac{\pi}{2} \right)^4 + \frac{7}{720} \left( x + \frac{\pi}{2} \right)^6 + \ldots
\] (111)

**Remark 1. (For those who are curious)** In fact one can show that for this problem, all \( a_n \) with \( n \) odd must be 0. Assume the contrary. Let \( a_n t^n \) be the lowest order term with odd power. As \( y'(0) = 0 \), \( n \geq 3 \), \( a_n \) must be determined by balancing the \( t^{n-2} \) term in

\[
y'' + (\sin t) y' - y = 0 \implies y'' + \left( t - \frac{t^3}{6} + \ldots \right) y' - y = 0.
\] (112)

Now there is no \( t^{n-2} \) term in \( y \) (since \( a_n t^n \) is the lowest order term with odd power). In the product, since the expansion of \( \sin t \) involves only odd powers of order at least 1, any \( t^{n-2} \) term must involve an even power of at most \( n - 3 \) in \( y' \). But such term can only come from differenting an odd power term of at most \( n - 2 \) in \( y \). Such term doesn’t exist (again, because \( a_n t^n \) is the lowest odd power term in \( y \)). So balancing \( t^{n-2} \) terms in the equation gives \( a_n n (n - 1) t^{n-2} = 0 \) and consequently \( a_n = 0 \). Contradiction.

Note that the above argument breaks down when \( a_1 \neq 0 \).

**Problem 13. (8.4 27)** Find at least first four nonzero terms of the power series expansion about \( x = 0 \) of a general solution to the given differential equation

\[
(1 - x^2) y'' - y' + y = \tan x.
\] (113)

**Solution.** We write \( y \) up to four terms

\[
y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots
\] (114)

and substitute into the equation. However we have to first expand \( \tan x \) to at least 4 terms at \( x = 0 \). We have

\[
\tan 0 = 0
\] (115)

\[
(tan x)' = \frac{1}{\cos^2 x} \implies (\tan x)'(0) = 1;
\] (116)

\[
(tan x)'' = 2 (\cos x)^{-3} \sin x \implies (\tan x)''(0) = 0;
\] (117)

\[
(tan x)''' = 6 (\cos x)^{-4} \sin^2 x + 2 (\cos x)^{-2} \implies (\tan x)'''(0) = 2.
\] (118)

So up to first four terms, we have

\[
\tan x = x + \frac{x^3}{3} + \ldots
\] (119)

Now substituting \( y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \) into the equation we have

\[
(1 - x^2) \left( 2 a_2 + 6 a_3 x + \cdots \right) - (a_1 + 2 a_2 x + 3 a_3 x^2 + \cdots) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = x + \frac{x^3}{3} + \ldots
\] (120)

Noticing that balancing the constant term would give us \( a_2 \) and balancing the \( x \) term would give us \( a_3 \), we only need to expand each term up to \( x \). So the equation becomes

\[
2 a_2 + 6 a_3 x + \cdots - (a_1 + 2 a_2 x + \cdots) + (a_0 + a_1 x + \cdots) = x + \cdots
\] (121)

which gives

\[
2 a_2 - a_1 + a_0 = 0; \quad 6 a_3 - 2 a_2 + a_1 = 1
\] (122)
Therefore
\[ a_2 = \frac{a_1 - a_0}{2}, \quad a_3 = \frac{1 - a_0}{6}. \] (123)

Both are nonzero.

The solution up to first four terms is then
\[ y = a_0 + a_1 x + \frac{a_1 - a_0}{2} x^2 + \frac{1 - a_0}{6} x^3 + \cdots \] (124)

Or written in the form \( C_1 y_1 + C_2 y_2 + y_p \):
\[ y = a_0 \left[ 1 - \frac{x^2}{2} + \frac{x^3}{6} + \cdots \right] + a_1 \left[ x + \frac{x^2}{2} + \cdots \right] + \left[ \frac{x^3}{6} + \cdots \right] \] (125)

**Problem 14. (8.4 28)** Find at least first four nonzero terms of the power series expansion about \( x = 0 \) of a general solution to the given differential equation.

\[ y'' - (\sin x) y = \cos x. \] (126)

**Solution.** Since we need first four terms we need to at least expand \( y \) up to four terms:
\[ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \] (127)

and substitute into the equation: \[ 2 a_2 + 6 a_3 x + \cdots = 1 \left[ 1 - \frac{x^2}{2} + \cdots \right] \]

Expanding the left hand side we obtain
\[ 2 a_2 + (6 a_3 - a_0) x + \cdots = 1 - \frac{x^2}{2} + \cdots \] (130)

This gives
\[ 2 a_2 = 1 \] (131)
\[ 6 a_3 - a_0 = 0 \] (132)

and we have \( y \) up to first four nonzero terms:
\[ y = a_0 + a_1 x + \frac{x^2}{2} + \frac{a_0}{6} x^3 + \cdots \] (133)

Note that to obtain \( a_2 \) and \( a_3 \) we only need to balance the constant term and the \( x \) term. So in fact we only need to expand everything in \(- (\sin x) y \) and \( \cos x \) up to \( x \), that is expand the equation to just
\[ 2 a_2 + 6 a_3 x + \cdots = 1 - \cdots. \] (128)

See that this leads to the same equations for \( a_2 \) and \( a_3 \).