Many examples here are taken from the textbook. The first number in ( ) refers to the problem number in the UA Custom edition, the second number in ( ) refers to the problem number in the 8th edition.

0. Review

To solve general 2nd order linear equations,
\[ a(t) y'' + b(t) y' + c(t) y = f(t). \]  

1. Guess one solution \( y_1 \). (Popular guesses: constants; exponentials; simple polynomials; \( \sin, \cos \))

2. Write the equation into standard form
\[ y'' + p(t) y' + q(t) y = 0 \]

and apply the “reduction of order” formula:
\[ y_2(t) = y_1(t) \int e^{-\int p(t) dt} \frac{dy_1^2}{y_1^2} \, dt \]

3. Apply variation of parameters formula
\[ y_p = v_1 y_1 + v_2 y_2 \]

with
\[ v_1 = \int \frac{-f(t) y_2(t)}{a(t) [y_1 y_2' - y_1' y_2]}, \quad v_2 = \int \frac{f(t) y_1(t)}{a(t) [y_1 y_2' - y_1' y_2]} \]

to obtain \( y_p \). Note that \( a(t) \) is not a constant anymore.

4. The general solution is then given by
\[ y = C_1 y_1 + C_2 y_2 + y_p. \]

Simplify if possible.

5. Check your solution if time allows.

Quiz: Solve
\[ t y'' - y' = 1. \]

(As \( t = 0 \) is a singular point, let’s just consider \( t > 0 \)).

Solution.

1. Guess one solution for
\[ t y'' - y' = 0. \]

It is clear that any constant is a solution. So let’s take \( y_1 = 1 \).

2. Use reduction of order to get \( y_2 \).

   a. Write the equation in standard form:
   \[ y'' - \frac{1}{t} y' = \frac{1}{t} \implies p(t) = -\frac{1}{t}. \]

   b. Compute
   \[ y_2 = y_1 \int e^{-\int \frac{1}{t} dt} = \int e^{\int \frac{1}{t} dt} = \int e^{\ln t} = \int t = \frac{t^2}{2}. \]

3. Use variation of parameters to get \( y_p \).
Compute
\[ a(t) \left[ y_1 y_2' - y_1' y_2 \right] = t \left[ 1 \left( \frac{t^2}{2} \right)' - \frac{t^2}{2} (1)' \right] = t^2. \] (11)

\[ v_1 = \int \frac{-f(t) y_2(t)}{a(t) [y_1 y_2' - y_1' y_2]} = -\int \frac{1 \cdot t^2}{2 t^2} = -\frac{t}{2}. \] (12)

\[ v_2 = \int \frac{f(t) y_1(t)}{a(t) [y_1 y_2' - y_1' y_2]} = \int \frac{1 \cdot 1}{t^2} = -\frac{1}{t} \] (13)

So
\[ y_p = \left( -\frac{t}{2} \right) (1) + \left( \frac{1}{t} \right) \frac{t^2}{2} = -t. \] (14)

4. The general solution is
\[ y = C_1 + C_2 t^2 - t \] (15)
which can be simplified to
\[ y = C_1 + C_2 t^2 - t. \] (16)

1. Basic Information

- **The Equation:**
  \[ a t^2 y'' + b t y' + c y = 0. \] (17)

Such equations are called “Cauchy-Euler equations” and they are as easy as equations of constant coefficients (In fact, Cauchy-Euler equations is just constant-coefficient equations in disguise, see Notes and Comments).

- **How to get general solution**
  - Idea: From general theory of 2nd order linear equations, we know that as soon as we figure out two linearly independent solutions \( y_1, y_2 \), the general solution is simply
  \[ C_1 y_1 + C_2 y_2. \] (18)

Recall that in studying constant-coefficient equations \( a y'' + b y' + c y = 0 \) we obtain \( y_1, y_2 \) through guess \( y = e^{rt} \). Here the idea is similar but the guess is different: \( y = t^r \).

  - Procedure:
    1. Write down the characteristic equation
       \[ a r^2 + (b - a) r + c = 0. \] (19)
       Solve it to get \( r_1, r_2 \).
    2. Three cases:
      - \( r_1 \neq r_2 \), both real:
        \[ y_1 = t^{r_1}, y_2 = t^{r_2}; \] (20)
      - \( r_1 = r_2 = r \), real.
        \[ y_1 = t^r, y_2 = t^r \ln t. \] (21)
      - \( r_{1,2} = \alpha \pm i \beta \).
        \[ y_1 = t^\alpha \cos (\beta \ln t), \quad y_2 = t^\alpha \sin (\beta \ln t). \] (22)
    3. Write down general solution.

  - Examples:

**Example 1.** Solve
\[ t^2 y'' + 7 t y' - 7 y = 0. \] (23)
Solution. Guided by the theory, we only need to find two linearly independent solutions. The key now is to realize the following property of $t^r$: $(t^r)^{(k)} t^k = C t^r$. Substitute $y = t^r$ into the equation, we have

$$0 = t^2 y'' + 7 t y' - 7 y = [r (r - 1) + 7 r - 7] t^r = 0 \iff r^2 + 6 r - 7 = 0 \implies r_1 = -7, r_2 = 1.$$  \hfill (24)

Thus the general solution is

$$y = c_1 t^{-7} + c_2 t.$$  \hfill (25)

**Example 2.** Solve

$$t^2 y'' - 3 t y' + 4 y = 0.$$  \hfill (26)

**Solution.** Substituting $y = t^r$ we have

$$r (r - 1) - 3 r + 4 = 0 \implies r^2 - 4 r + 4 = 0 \implies r_1 = r_2 = 2.$$  \hfill (27)

This is double root so the general solution is given by

$$y = c_1 t^2 + c_2 t^2 \ln t.$$  \hfill (28)

**Example 3.** Solve

$$y'' - \frac{1}{t} y' + \frac{5}{t^2} y = 0.$$  \hfill (29)

**Solution.** Multiply both sides by $t^2$:

$$t^2 y'' - t y' + 5 y = 0.$$  \hfill (30)

Substituting $y = t^r$ gives

$$r (r - 1) - r + 5 = 0 \implies r^2 - 2 r + 5 = 0 \implies r_1 = 1 + 2 i, r_2 = 1 - 2 i.$$  \hfill (31)

The general solution is then given by

$$y = c_1 t \cos (2 \ln t) + c_2 t \sin (2 \ln t).$$  \hfill (32)

• Relations to constant-coefficient equations. The formulas

- $r_1 \neq r_2$, both real:
  $$y_1 = t^{r_1}, y_2 = t^{r_2};$$  \hfill (33)

- $r_1 = r_2 = r$, real.
  $$y_1 = t^r, y_2 = t^r \ln t.$$  \hfill (34)

- $r_{1,2} = \alpha \pm i \beta$.
  $$y_1 = t^\alpha \cos (\beta \ln t), \quad y_2 = t^\alpha \sin (\beta \ln t).$$  \hfill (35)

above looks similar to our theory for linear constant-coefficient equations. This similarity becomes more striking if we introduce a new variable $x = \ln t$: The three cases become

- Two distinct roots $\implies e^{r_1 x}, e^{r_2 x};$
- One double root $\implies e^{rx}, xe^{rx};$
- Complex roots $\implies e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x.$

This is no coincidence! In fact, setting $x = \ln t$ gives

$$y' = \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = -t^{-1} \frac{dy}{dx}, \quad y' = \frac{d^2 y}{dt^2} = \frac{d}{dx} \left[ t^{-1} \frac{dy}{dx} \right] \frac{dx}{dt} = t^{-2} \frac{d^2 y}{dx} - t^{-2} \frac{dy}{dx}.$$  \hfill (36)

Substituting into the equation

$$0 = a t^2 y'' + bt y' + cy = a \frac{d^2 y}{dx^2} + (b - a) \frac{dy}{dx} + cy.$$  \hfill (37)
Thus we have transformed the Euler-Cauchy equation into a constant-coefficient equation. Furthermore, the auxiliary equation for this equation is

\[ ar^2 + (b-a) r + c = ar(r-1) + br + c \] (38)

which is exactly the characteristic equation of the Cauchy-Euler equation!

- **How to check solutions**
  - Note: Check solutions, especially in the 3rd case, may involve so much calculation that it becomes not worthwhile. Instead, make sure you write down the correct characteristic equation and solve it correctly.

2. **Things to be Careful/Tricky Issues**

- Fail to see that the equation is Cauchy-Euler. (This is the hottest mistake in 201!!)
- After finding \( r_1, r_2 \), write \( e^{r_1 t}, e^{r_2 t} \) instead of \( t^{r_1}, t^{r_2} \). (This mistake is also very popular.)

3. **More Examples**

**Example 4.** Solve the following initial value problem for the Cauchy-Euler equation

\[ t^2 y'' - 4 ty' + 4 y = 0; \quad y(1) = -2, \quad y'(1) = -11. \] (39)

**Solution.** Substituting \( y = t^r \) gives

\[ r(r-1) - 4r + 4 = 0 \iff r^2 - 5r + 4 = 0 \iff r_1 = 4, \quad r_2 = 1. \] (40)

Thus the general solution is given by

\[ y(t) = c_1 t^4 + c_2 t. \] (41)

Using the initial values, we have

\[ -2 = y(1) = c_1 + c_2; \quad -11 = y'(1) = 4c_1 + c_2. \] (42)

Solving this we reach

\[ c_1 = -3, \quad c_2 = 1. \] (43)

Thus the solution to the initial value problem is given by

\[ y(t) = -3t^4 + t. \] (44)

- What happens if \( t < 0 \)?

**Example 5.** (NA: 4.7 15) Solve for \( t < 0 \)

\[ y'' - \frac{1}{t} y' + \frac{5}{t^2} = 0. \] (45)

**Solution.** Let \( x = -t \). Then \( x > 0 \). We have

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = -\frac{dy}{dx}, \quad \frac{d^2y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = -\frac{d}{dx} \left( \frac{dy}{dx} \right) \cdot \frac{dx}{dt} = \frac{d^2y}{dx^2}. \] (46)

So if we use \( x \) instead of \( t \) as the variable, the equation (with unknown \( y \) and variable \( x \)) reads

\[
\frac{d^2y}{dx^2} - \frac{1}{x} \frac{dy}{dx} + \frac{5}{x^2} = 0. \] (47)

It is still Cauchy-Euler, with \( a = 1, b = -1, c = 5 \). We write down characteristic equation

\[ r^2 - 2r + 5 = 0 \iff r_{1,2} = 1 \pm 2i. \] (48)
So the solution reads
\[ y(x) = C_1 x \cos (2 \ln x) + C_2 x \sin (2 \ln x). \] (49)

Back to \( t \):
\[ y(t) = C_1 (-t) \cos (2 \ln (-t)) + C_2 (-t) \sin (2 \ln (-t)). \] (50)

Which can be written as
\[ y(t) = C_1 |t| \cos (2 \ln |t|) + C_2 |t| \sin (2 \ln |t|). \] (51)

**Remark 6.** In fact, we can solve Cauchy-Euler for \( t \neq 0 \) as

- \( r_1 \neq r_2 \), both real:
  \[ y_1 = |t|^r_1, \quad y_2 = |t|^r_2; \] (52)
- \( r_1 = r_2 = r \), real:
  \[ y_1 = |t|^r, \quad y_2 = |t|^r \ln |t|. \] (53)
- \( r_{1,2} = \alpha \pm i \beta \).
  \[ y_1 = |t|^\alpha \cos (\beta \ln |t|), \quad y_2 = |t|^\alpha \sin (\beta \ln |t|). \] (54)

- What if it’s not \( t \)?

**Example 7. (4.7 21; 4.7 21)** Solve
\[ (t - 2)^2 y'' - 7 (t - 2) y' + 7 y = 0, \quad t > 2. \] (55)

**Solution.** It is clear that we should introduce \( x = t - 2 \). Now chain rule gives
\[ \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}, \quad \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} \frac{dx}{dt}^2. \] (56)

The equation becomes
\[ x^2 \frac{d^2y}{dx^2} - 7 x \frac{dy}{dx} + 7 y = 0, \quad x > 0. \] (57)

This can be easily solved:
\[ y(x) = C_1 x + C_2 x^7. \] (58)

Back to \( t \):
\[ y(t) = C_1 (t - 2) + C_2 (t - 2)^7. \] (59)

- Nonhomogeneous problem.

  - There are two ways to attack nonhomogeneous problem for Cauchy-Euler equations.
    1. Introduce \( x = \ln t \), transform it to constant-coefficient case, then apply undetermined coefficients, or variation of parameters;
    2. Apply variation of parameters directly.

  - Examples.

**Example 8.** Solve
\[ t^2 y'' - 4 t y' + 4 y = t^2. \] (60)

**Solution 1 \((x = \ln t)\).**

We know that setting \( x = \ln t \) transforms
\[ a t^2 y'' + b t y' + c y \text{ to } \frac{d^2y}{dx^2} + (b - a) \frac{dy}{dx} + c y. \] (61)

So the equation for \( x \) is \((x = \ln t \Rightarrow t = e^x)\):
\[ \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 4 y = e^{2x}. \] (62)
We see that this equation is eligible for undetermined coefficients (keep in mind that whenever undetermined coefficients applies, it is more efficient than variation of parameters).

To solve it we first get \( y_1, y_2 \) by solving
\[
\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 4y = 0
\]
which gives
\[
r_1 = 4, r_2 = 1; \quad y_1 = e^{4x}, y_2 = e^x.
\]
(63)

Now guess
\[
y_p = A x^s e^{2x}.
\]
(64)

We have \( s = 0 \) because 2 does not appear in the root list \( r_1 = 4, r_2 = 1 \).

Substitute \( y_p = A x^s e^{2x} \) into the equation we get
\[
\frac{d^2y}{dx^2} = 4A e^{2x}, \quad \frac{dy}{dx} = 2A e^{2x} \implies 4A - 10A + 4A = 1 \implies A = -\frac{1}{2}
\]
(66)

So \( y_p = -\frac{e^{2x}}{x} \). The general solution (in \( x \)) is then
\[
y = C_1 e^{4x} + C_2 e^x - \frac{1}{2} e^{2x}.
\]
(67)

Back to \( t \): (Replace every \( x \) by \( \ln t \)):
\[
y(t) = C_1 t^4 + C_2 t - \frac{1}{2} t^2.
\]
(68)

**Solution 2 (Direct application of variation of parameters).**

The equation is Cauchy Euler so we can solve the homogeneous equation
\[
t^2 y'' - 4 ty' + 4y = 0
\]
as follows:
\[
r(r - 1) - 4r + 4 = 0 \implies r^2 - 5r + 4 = 0 \implies r_1 = 4, r_2 = 1.
\]
(70)

So
\[
y_1 = t^4, y_2 = t.
\]
(71)

Now calculate:
\[
a(t) [y_1 y_2' - y_1'y_2] = t^2 [t^4 - 4 t^3 t] = -3 t^6.
\]
(72)

Thus
\[
v_1 = \int \frac{-f(t) y_2(t)}{a(t) [y_1 y_2' - y_1'y_2]} = \int \frac{-t^2 t}{-3 t^6} = \int \frac{1}{3 t^4} = -\frac{1}{6} t^{-2}.
\]
(73)

\[
v_2 = \int \frac{f(t) y_1(t)}{a(t) [y_1 y_2' - y_1'y_2]} = \int \frac{2 t^4 t^4}{-3 t^6} = -\frac{t}{3}.
\]
(74)

Therefore
\[
y_p = v_1 y_1 + v_2 y_2 = \left(-\frac{1}{6} t^{-2} \right) t^4 + \left(-\frac{t}{3} \right) t = -\frac{t^2}{2}.
\]
(75)

The general solution is then given by
\[
y(t) = C_1 t^4 + C_2 t - \frac{1}{2} t^2.
\]
(76)

**Example 9. (4.7 41; 4.7 41)** Solve
\[
t^2 z'' + tz' + 9z = -\tan(3 \ln t).
\]
(77)

**Solution.** First solve the homogeneous equation
\[
t^2 z'' + tz' + 9z = 0.
\]
(78)
It is Cauchy-Euler. So first solve the characteristic equation:

\[ r(r - 1) + r + 9 = 0 \Rightarrow r_{1,2} = \pm 3. \]  

(79)

So we have

\[ y_1 = \cos (3 \ln t), \quad y_2 = \sin (3 \ln t). \]  

(80)

Next we use variation of parameters to obtain \( y_p \). First calculate:

\[
a [y_1 y_2' - y_1' y_2] = t^2 \left[ \cos (3 \ln t) \left( \frac{3}{t} \sin (3 \ln t) \right) - \left( \frac{3}{t} \cos (3 \ln t) \right) \sin (3 \ln t) \right] \]
\[
= t^2 \frac{3}{t} = 3t. \]  

(81)

Now we have

\[
v_1 = - \int \frac{-\tan (3 \ln t) \sin (3 \ln t)}{3t} \, dt \]
\[
= \frac{1}{3} \int \frac{\sin^2 (3 \ln t)}{\cos (3 \ln t)} \, d\ln t. \]  

(82)

Letting \( x = \ln t \), we compute

\[
\int \frac{\sin^2 (3x)}{\cos (3x)} \, dx = \int \frac{dx}{\cos (3x)} - \int \cos (3x) \, dx
\]
\[
= \int (\cos^2 (3x) - 1) \sin (3x) \, dx - \frac{1}{3} \sin (3x)
\]
\[
= \frac{1}{3} \left( \int \frac{1}{1 - \sin^2 (3x)} \, dsin (3x) - \frac{1}{3} \sin (3x) \right)
\]
\[
= \frac{1}{6} \left( \ln |1 + \sin (3x)| - \ln |1 - \sin (3x)| - \frac{1}{3} \sin (3x) \right). \]  

(83)

Back to \( t \):

\[
v_1 = \frac{1}{3} \left\{ \frac{1}{6} \ln |1 + \sin (3 \ln t)| - \ln |1 - \sin (3 \ln t)| - \frac{1}{3} \sin (3 \ln t) \right\}. \]  

(84)

On the other hand

\[
v_2 = \int \frac{(-\tan (3 \ln t)) \cos (3 \ln t)}{3t} \, dt
\]
\[
= - \int \frac{\sin (3 \ln t)}{3t} \, dt
\]
\[
= - \frac{1}{3} \int \sin (3 \ln t) \, d\ln t
\]
\[
= \frac{1}{9} \cos (3 \ln t); \]  

(85)

Putting things together we have

\[
y_p = \frac{1}{18} \left[ \ln |1 + \sin (3 \ln t)| - \ln |1 - \sin (3 \ln t)| \right] \cos (3 \ln t)
\]
\[
- \frac{1}{9} \sin (3 \ln t) \cos (3 \ln t)
\]
\[
+ \frac{1}{9} \cos (3 \ln t) \sin (3 \ln t)
\]
\[
= \frac{1}{18} \left[ \ln |1 + \sin (3 \ln t)| - \ln |1 - \sin (3 \ln t)| \right] \cos (3 \ln t) \]  

(86)

Thus the solution is

\[
y = C_1 \cos (3 \ln t) + C_2 \sin (3 \ln t) + \frac{1}{18} \left[ \ln |1 + \sin (3 \ln t)| - \ln |1 - \sin (3 \ln t)| \right] \cos (3 \ln t). \]  

(87)
Remark 10. For such problems sometimes it is more efficient to let \( x = \ln t \) and transform the equation (See Notes and Comments). For example for the above problem, if we let \( x = \ln t \), then the equation becomes

\[
\frac{d^2 y}{dx^2} + 9 y = -\tan(3x).
\]  

(88)

This can be solved by variation of parameters, to obtain

\[
y = C_1 \cos(3x) + C_2 \sin(3x) + \frac{1}{18} \ln|1 + \sin(3x)| - \ln|1 - \sin(3x)| \cos(3x).
\]  

(89)

Replace \( x \) by \( \ln t \) we get our solution.

4. Notes and Comments

- Why \( b - a \)?
  
  To remember it, just remember that our guess is \( y = t^r \). Substituting this into the equation we reach

\[
ar (r - 1) + br + c = 0
\]  

(90)

which is exactly

\[
ar^2 + (b - a) r + c = 0.
\]  

(91)

- Why do we need \( t > 0 \)?
  
  Note that, unlike \( e^{rt} \), \( t^r \) is singular (meaning: either it’s infinity, or its certain order of derivative is infinity) at \( t = 0 \).

  More accurately, when writing the Cauchy-Euler equation in standard form

\[
y'' + \frac{b/a}{t} y' + \frac{c/a}{t^2} y = 0.
\]  

(92)

We see that \( p(t) = \frac{b/a}{t} \) and \( q(t) = \frac{c/a}{t^2} \) are singular at 0. This is an indication that good theories (solution exists, solution is unique, solution is smooth...) break down when the interval for \( t \) contains 0. Therefore to make things simple we either work in \( t > 0 \) or in \( t < 0 \), to avoid containing \( t = 0 \).