0. Review

The method of undetermined coefficients allow us to solve equations of the type
\[ a y'' + b y' + c y = f(t) \]
with \( f(t) \) of special forms:
\[ f(t) = P_m(t) e^{rt} \text{ or } f(t) = P_m(t) e^{at} \cos{\beta t} + Q_n(t) e^{at} \sin{\beta t} \text{ or Sum of such terms} \]

It should be emphasized again that the method only works when
1. The equation is constant-coefficient (\( a, b, c \) are constants)
2. \( f(t) \) is of special form.

To procedure goes:
1. Solve the characteristic equation of the homogeneous equation, write down the list of roots.
   Organize complex roots in conjugate pairs.
2. Guess \( y_p \) according to:
   - If \( f(t) = C P_m(t) e^{rt} \), then
     \[ y_p = t^s (A_m t^m + \ldots + A_1 t + A_0) e^{rt} \]
     with \( s \) = the number of times \( r \) appears in the list of roots \( r_1, r_2 \).
   - If \( f(t) = P_m(t) e^{at} \cos{\beta t} + Q_n(t) e^{at} \sin{\beta t} \), then
     \[ y_p = t^s (A_k t^k + \ldots + A_1 t + A_0) e^{at} \cos{\beta t} + t^s (B_k t^k + \ldots + B_0) e^{at} \sin{\beta t} \]
     with \( s \) = the number of times the pair \( (a \pm \beta i) \) appears in the list of roots \( r_1, r_2 \), and
     \[ k = \max \{m, n\} \] that is the bigger number of \( m, n \).
   - If \( f(t) = f_1 + \ldots + f_k \) with each \( f_i \) of one of the above form, then find \( y_{pi} \) for each
     \[ a y'' + b y' + c y = f_i(t) \]
     and set \( y_p = y_{p_1} + \ldots + y_{pk} \).
3. Substitute \( y_p \) into the equation to find all the constants.
4. Write down the final answer.

Quiz:
\[ y'' - y' = \cos{t} + 1, \quad y(0) = y'(0) = 0. \]

Solution. First notice that \( f(t) = \cos{t} + 1 \) is of the form \( f = f_1 + f_2 \) with
\[ f_1 = \cos{t}, \quad f_2 = 1 \]
1. Solve
\[ y'' - y' = 0. \]
   Characteristic equation \( r^2 - r = 0 \implies r_1 = 0, r_2 = 1 \). So \( y_1 = 1, y_2 = e^t \).
2. Find $y_p$.

- $y_{p1}$. $y_{p1}$ is a particular solution to
  \[ y'' - y' = \cos t. \]  \(9\)

  Writing
  \[ \cos t = t^0 e^{0t} \cos t \]  \(10\)

  we see that
  \[ y_{p1} = t^s A \cos t + t^s B \sin t. \]  \(11\)

  As $\alpha \pm i \beta = 0 \pm i = \pm i$ does not appear in $r_1, r_2$, we have $s = 0$. So
  \[ y_{p1} = A \cos t + B \sin t. \]  \(12\)

  To get $A, B$, substitute the above into the equation:
  \[ \cos t = y_{p1}'' - y_{p1}' = -A \cos t - B \sin t + A \sin t - B \cos t = (A - B) \sin t - (A + B) \cos t \]  \(13\)

  So
  \[ A - B = 0, -(A + B) = 1 \implies A = B = -\frac{1}{2} \]  \(14\)

  So
  \[ y_{p1} = -\frac{1}{2} (\sin t + \cos t). \]  \(15\)

- $y_{p2}$. $y_{p2}$ is a particular solution to
  \[ y'' - y' = 1. \]  \(16\)

  Writing
  \[ 1 = t^0 e^{0t} \]  \(17\)

  we see that
  \[ y_{p2} = t^s A. \]  \(18\)

  To determine $s$ we check the number of times $0$ appears in the list $r_1, r_2$: We get $s = 1$. So
  \[ y_{p2} = A t. \]  \(19\)

  Substituting this into the equation we get $A = -1$.

  So
  \[ y_{p} = -\frac{1}{2} (\sin t + \cos t) - t. \]  \(20\)

3. The general solution is
  \[ y = C_1 + C_2 e^t - \frac{1}{2} (\sin t + \cos t) - t. \]  \(21\)

4. To solve the initial value problem, we prepare
  \[ y' = C_2 e^t - \frac{1}{2} (\cos t - \sin t) - 1. \]  \(22\)

  So
  \[ y(0) = 0 \implies C_1 + C_2 - \frac{1}{2} = 0 \]  \(23\)

  \[ y'(0) = 0 \implies C_2 - \frac{3}{2} = 0. \]  \(24\)

  So
  \[ C_1 = -1, C_2 = \frac{3}{2}. \]  \(25\)

5. Summarizing, we have the final answer:
  \[ y = -1 + \frac{3}{2} e^t - \frac{1}{2} (\sin t + \cos t) - t. \]  \(26\)
1. Basic Information

- The Equation:
  Still
  \[ a \ y'' + b \ y' + c \ y = f(t) \]  
  (27)

  But with \( f(t) \) subject to no restriction.

  **Remark 1.** We will see in the next lecture that the following method actually works for the more general case
  \[ a(t) \ y'' + b(t) \ y' + c(t) \ y = f(t) \]  
  (28)

  (Yes this is the general 2nd order linear equation).

- How to get general solution
  1. Solve the homogeneous equation, get \( y_1, y_2 \).
  2. Calculate a particular solution using the following formula:
     \[ y_p = v_1 \ y_1 + v_2 \ y_2 \]  
     (29)

     with
     \[ v_1 = \int \frac{-f(t) \ y_2(t)}{a \ [y_1 \ y_2 - y_1' \ y_2]}, \quad v_2 = \int \frac{f(t) \ y_1(t)}{a \ [y_1 \ y_2 - y_1' \ y_2]} \]  
     (30)

     See “Notes and Comments” for derivation of these two formulas.

  **Remark 2.** Note that \( y_1 \ y_2' - y_1' \ y_2 \) is the Wronskian of the two solutions \( y_1, y_2 \):

  \[ W[y_1, y_2](t) = \text{det} \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2. \]  
  (31)

  Further note that \( y_1 y_2' - y_1' y_2 \) is guarantee to be nonzero due to the following result:

  If \( y_1, y_2 \) are solution to the same 2nd order homogeneous linear equation, then \( y_1, y_2 \) linearly independent \( \iff W[y_1, y_2] \neq 0 \).  
  (32)

  3. Write down the final answer
     \[ y = C_1 \ y_1 + C_2 \ y_2 + y_p = C_1 \ y_1 + C_2 \ y_2 + v_1 \ y_1 + v_2 \ y_2. \]  
     (33)

  4. Simplify.\textsuperscript{1}
    - Example: Solve
      \[ y'' - 2 \ y' + y = \frac{2 e^t}{t}. \]  
      (34)

  **Solution.** First we notice that the right hand side makes it impossible to apply undetermined coefficients.

  Now we apply variation of parameters.

  1. Solve the homogeneous equation
     \[ y'' - 2 \ y' + y = 0 \implies y_1 = e^t, \ y_2 = t \ e^t. \]  
     (35)

  2. Calculate:
     \[ a \ [y_1 \ y_2' - y_1' \ y_2] = 1 \ [e^t (e^t + t \ e^t) - e^t t \ e^t] = e^{2t}. \]  
     (36)

\textsuperscript{1} It may happen that \( v_1, v_2 \) contain some constants, say \( v_1 = 6 \ t + 5 \). Then the 5 \( y_1 \) can be combined into the \( C_1 \ y_1 \) term.
As \( f(t) = \frac{2e^t}{t} \) we have

\[
v_1 = \int \frac{-f(t) y_2(t)}{a (y_1 y_2' - y_1' y_2)} \, dt = \int -\left(\frac{2e^t}{t}\right) \frac{e^t}{e^{2t}} \, dt = -2 = -2t. \tag{37}
\]

\[
v_2 = \int \frac{f(t) y_1(t)}{a (y_1 y_2' - y_1' y_2)} \, dt = \int \frac{\left(\frac{2e^t}{t}\right) e^t}{e^{2t}} \, dt = \int \frac{2}{t} = 2\ln|t|. \tag{38}
\]

3. The general solution is then

\[y = C_1 y_1 + C_2 y_2 + y_p = C_1 y_1 + C_2 y_2 + v_1 y_1 + v_2 y_2 = C_1 e^t + C_2 t e^t - 2 t e^t + 2\ln|t| t e^t. \tag{39}\]

4. Simplify:

\[y = C_1 e^t + C_2 t e^t - 2 t e^t + 2\ln|t| t e^t
= C_1 e^t + (C_2 - 2) t e^t + 2\ln|t| t e^t. \tag{40}\]

As \( C_2 \) stands for an arbitrary constant, so does \( C_2 - 2 \). So we can write the general solution as

\[y = C_1 e^t + C_2 t e^t + 2\ln|t| t e^t. \tag{41}\]

• How to solve initial value problem (IVP)
  - Nothing new.
  - Example: Solve
    \[y'' - 2y' + y = \frac{2e^t}{t}, \quad y(1) = 0, y'(1) = 1. \tag{42}\]

**Solution.** As usual, an initial value problem is solved by first getting the general solution, then apply the initial conditions. The general solution is already obtained as

\[y = C_1 e^t + C_2 t e^t + 2\ln|t| t e^t. \tag{43}\]

To apply the initial conditions, we prepare

\[y' = C_1 e^t + C_2 t e^t + C_2 t e^t + 2 e^t + 2 e^t + 2\ln|t| e^t + 2\ln|t| t e^t. \tag{44}\]

Now the initial conditions lead to

\[y(1) = 0 \implies C_1 e + C_2 e = 0. \tag{45}\]

\[y'(1) = 0 \implies C_1 + 2 C_2 = 1. \tag{46}\]

Solving this we obtain

\[C_1 = 1, C_2 = -1. \tag{47}\]

So the answer is

\[y = e^t - t e^t + 2\ln|t| t e^t. \tag{48}\]

• How to check solutions
  - Nothing new: Substitute \( y = C_1 e^t + C_2 t e^t + 2\ln|t| t e^t \) into the equation.

2. THINGS TO BE CAREFUL/TRICKY ISSUES

See “Common Mistakes” for examples.

• Fail to be consistent in which is \( y_1 \) and which is \( y_2 \).
• Forget to simplify.

3. More Examples

*Example: (4.6 1; 4.6 2)* Find the general solution to the differential equation

\[ y'' + 4y = \tan 2t. \]  

**Solution.** We need to do two things: finding general solution to the homogeneous equation, and finding one particular solution to the nonhomogeneous equation.

- **General solution to the homogeneous problem.** The auxiliary equation is

  \[ r^2 + 4 = 0 \rightarrow r_1 = 2i, r_2 = -2i \implies z_1 = e^{2it} = \cos 2t + i \sin 2t, \quad z_2 = \cos 2t - i \sin 2t. \]  

  Thus the general solution to the homogeneous equation is given by

  \[ C_1 \cos 2t + C_2 \sin 2t. \]

- **Particular solution to the nonhomogeneous problem.** As the right hand side is \( \tan 2t \), it is not possible to use the method of undetermined coefficients. We use variation of parameters instead. We have

  \[ y_1 = \cos 2t, \quad y_2 = \sin 2t, \quad f(t) = \tan 2t = \frac{\sin 2t}{\cos 2t}, \quad a = 1. \]

  We compute

  \[ y_1 y_2' - y_1' y_2 = 2. \]

  Thus the equations for \( v_1, v_2 \) are

  \[ v_1 = \int \frac{-f(t) y_2(t)}{a [y_1 y_2' - y_1' y_2]} \, dt = -\frac{1}{2} \int \frac{\sin^2 2t}{\cos 2t} \, dt \quad v_2 = \int \frac{f(t) y_1(t)}{a [y_1 y_2' - y_1' y_2]} \, dt = \frac{1}{2} \int \sin 2t. \]

  Integrating, we have

  \[ v_1(t) = -\frac{1}{2} \int \frac{\sin^2 2t}{\cos 2t} \, dt \]

  \[ = -\frac{1}{2} \int \frac{1 - \cos 2t}{\cos 2t} \, dt \]

  \[ = -\frac{1}{2} \int \frac{1}{\cos 2t} \, dt + \frac{1}{2} \int \cos 2t \, dt. \]  

  We evaluate

  \[ \frac{1}{2} \int \cos 2t \, dt = \frac{1}{4} \sin 2t. \]

  \[ -\frac{1}{2} \int \frac{1}{\cos 2t} \, dt = -\frac{1}{2} \int \frac{\cos 2t}{\cos^2 2t} \, dt \]

  \[ = -\frac{1}{4} \int \frac{d\sin 2t}{1 - \sin^2 2t} \]

  \[ = -\frac{1}{8} \left[ \int \frac{d\sin 2t}{1 - \sin^2 2t} + \int \frac{d\sin 2t}{1 + \sin 2t} \right] \]

  \[ = -\frac{1}{8} [\ln |1 + \sin 2t| - \ln |1 - \sin 2t|]. \]

  Thus

  \[ v_1(t) = \int \frac{\sin^2 2t}{\cos 2t} \, dt = -\frac{1}{8} [\ln |1 + \sin 2t| - \ln |1 - \sin 2t|] + \frac{1}{4} \sin 2t. \]  

  On the other hand,

  \[ v_2(t) = \frac{1}{2} \int \sin 2t \, dt = -\frac{1}{4} \cos 2t. \]
Putting things together, we have
\[
y_p(t) = v_1 y_1 + v_2 y_2
= \left[ -\frac{1}{8} \ln |1 + \sin 2t| - \ln |1 - \sin 2t| + \frac{1}{4} \sin 2t \right] \cos 2t
- \frac{1}{4} \cos 2t \sin 2t
= -\frac{1}{8} \left[ \ln |1 + \sin 2t| - \ln |1 - \sin 2t| \right] \cos 2t. \tag{60}
\]

Summarizing, we have
\[
y = C_1 \cos 2t + C_2 \sin 2t - \frac{1}{8} \left[ \ln |1 + \sin 2t| - \ln |1 - \sin 2t| \right] \cos 2t. \tag{61}
\]

- Usually, when the method of “undetermined coefficients” applies, it will yield the answer faster than variation of parameters. So sometimes it is a good idea to combine the two methods (Thanks to linearity!).

**Example 3.** Solve
\[
y'' + 4y = \tan 2t + e^{3t} \tag{62}
\]

**Solution.**

First solve the homogeneous equation:
\[
y'' + 4y = 0 \implies r_{1,2} = \pm 2i \implies C_1 \cos 2t + C_2 \sin 2t. \tag{63}
\]

To find \(y_p\) we notice that \(\tan 2t\) is the “bad term” while \(e^{3t} - 1\) fits the framework of undetermined coefficients. So we break things up:
\[
y_p = y_{p1} + y_{p2} \tag{64}
\]

with
\[
y_{p1}'' + 4y_{p1} = \tan 2t \tag{65}
\]
\[
y_{p2}'' + 4y_{p2} = e^{3t} \tag{66}
\]

- Get \(y_{p1}\): We have already done that in the last example:
\[
y_{p1} = -\frac{1}{8} \left[ \ln |1 + \sin 2t| - \ln |1 - \sin 2t| \right] \cos 2t. \tag{67}
\]

- Get \(y_{p2}\): We have
\[
y_{p2} = t^s A e^{3t}. \tag{68}
\]

To determine \(s\) check how many times 3 appears in the roots list \(\pm 2i\): zero. So \(s = 0\) and \(y_{p2} = A e^{3t}\). Substituting into the equation gives
\[
9A e^{3t} + 4A e^{3t} = e^{3t} \implies A = \frac{1}{13}. \tag{69}
\]

So
\[
y_{p2} = \frac{1}{13} e^{3t}. \tag{70}
\]

Summarizing, we have
\[
y_p = -\frac{1}{8} \left[ \ln |1 + \sin 2t| - \ln |1 - \sin 2t| \right] \cos 2t + \frac{1}{13} e^{3t} \tag{71}
\]

and the general solution is then
\[
y = C_1 \cos 2t + C_2 \sin 2t - \frac{1}{8} \left[ \ln |1 + \sin 2t| - \ln |1 - \sin 2t| \right] \cos 2t + \frac{1}{13} e^{3t}. \tag{72}
\]

To check solution,

- Check that \(\cos 2t, \sin 2t\) indeed solves the homogeneous equation \(y'' + 4y = 0\);
Deriving the formulas

2. Recall from linear algebra that the system

\[
\begin{align*}
v_1 &= \int \frac{-f(t) y_2(t)}{a [y_1 y'_2 - y'_1 y_2]}, \\
v_2 &= \int \frac{f(t) y_1(t)}{a [y_1 y'_2 - y'_1 y_2]}
\end{align*}
\]  

(73)

Start from

\[
y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t).
\]  

(74)

Substituting this into the nonhomogeneous equation, we obtain

\[
f(t) = a y'' + b y' + c y
\]

\[
= a [v_1 y_1 + v_2 y_2]'' + b [v_1 y_1 + v_2 y_2]' + c [v_1 y_1 + v_2 y_2]
\]

\[
= a [v''_1 y_1 + 2 v'_1 y'_1 + v_1 y''_2 + 2 v'_2 y'_2 + v_2 y''_2 + 2 v'_2 y'_2]
\]

\[
+ b [v'_1 y_1 + v_1 y'_1 + v'_2 y_2 + v_2 y'_2] + c [v_1 y_1 + v_2 y_2]
\]

\[
= v_1 [a y'' + b y' + c y] + v_2 [a y'' + b y' + c y]
\]

\[
+ a [v'_1 y_1 + 2 v'_1 y_2 + 2 v_2 y'_2 + 2 v'_2 y'_2] + b [v'_1 y_1 + v'_2 y_2] + b [v'_1 y_1 + v'_2 y_2] + b [v'_1 y_1 + v'_2 y_2].
\]  

(75)

Thus as long as we can find \(v_1, v_2\) such that

\[
a [v''_1 y_1 + 2 v'_1 y'_1 + v''_2 y_2 + 2 v'_2 y'_2] + b [v'_1 y_1 + v'_2 y_2] = f(t),
\]

(76)

we are done.

This equation is quite complicated. We need to simplify it. Note that we have two unknowns \(v_1, v_2\) and just one equation, it is likely that we can require \(v_1, v_2\) to satisfy another equation without losing existence of solutions.

One good choice is requiring

\[
v'_1 y_1 + v'_2 y_2 = 0.
\]

(77)

Note that this not only makes the term containing \(b\) vanish, but also greatly simplify the first term through

\[0 = (v'_1 y_1 + v'_2 y_2)' = v''_1 y_1 + v'_1 y'_1 + v''_2 y_2 + v'_2 y'_2.
\]  

(78)

Thus our equations for \(v_1, v_2\) are

\[
a [v'_1 y'_1 + v'_2 y'_2] = f(t)
\]

(79)

\[
v'_1 y_1 + v'_2 y_2 = 0.
\]

(80)

Rewrite this system to

\[
y'_1 v'_1 + y'_2 v'_2 = f(t)/a
\]

(81)

\[
y'_1 v'_1 + y'_2 v'_2 = 0.
\]

(82)

This system admits a unique solution when \(y_1, y_2\) are linearly independent.\(^2\) The solution is given by

\[
v'_1 = \frac{-f(t) y_2(t)}{a [y_1 y'_2 - y'_1 y_2]}, \quad v'_2 = \frac{f(t) y_1(t)}{a [y_1 y'_2 - y'_1 y_2]}.
\]

(87)

---

2. Recall from linear algebra that the system

\[
a x + b y = e
\]

\[
c x + d y = f
\]

has a unique solution for all \(e, f\) if and only if

\[
\text{det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0.
\]

(85)
Then $v_1, v_2$ can be obtained from integrating the above.

- Comparison of the two methods:

<table>
<thead>
<tr>
<th></th>
<th>Undetermined Coefficients</th>
<th>Variation of Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Applicable to</td>
<td>Some $f$</td>
<td>All $f$</td>
</tr>
<tr>
<td>Need to solve</td>
<td>$a y'' + b y' + c y = 0$</td>
<td>Almost</td>
</tr>
<tr>
<td>$f$</td>
<td></td>
<td>Yes</td>
</tr>
<tr>
<td>Complexity of calculation</td>
<td>Low</td>
<td>High</td>
</tr>
<tr>
<td>Extension to variable coefficient equations?</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table 1. Solving $a y'' + b y' + c y = f(t)$, UC vs. VP.

- For equations with constant coefficients, we will introduce a third method, the method of Laplace transform, which is in some sense in between the current two methods: Simpler calculation than variation of parameters, but more complicated than undetermined coefficients, etc.

- Although variation of parameters can be extended to variable coefficient equations, in practice it is rarely used for the following reasons:
  - There is no good method getting formulas for solutions $y_1, y_2$ of the homogeneous equation $a(t) y'' + b(t) y' + c(t) y = 0$. Such formulas may not even exist;
  - The “series” method (Chapter 8) can get $y_1, y_2$ in the form of infinite sums. But they are hard to use in variation of parameters. Furthermore the same “series” method can get $y_p$ directly without much more difficulty than getting $y_1, y_2$ anyway.

---

When this is the case, the unique solution is given by Cramer’s rule

\[
x = \frac{\det \begin{pmatrix} e & b \\ f & d \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}, \quad y = \frac{\det \begin{pmatrix} a & e \\ b & f \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}.
\]

(86)