CALCULUS PROBLEMS IN PUTNAM EXAM

Calculus problems are by far the most common type of problems in Putnam. The techniques involved are way too diversified to classify. I will use examples to introduce some important techniques.

Example 0.1 (2000 A-1). Let $A$ be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

This is probably one of the easiest Putnam problem: $\inf \sum_{j=0}^{\infty} x_j^2 = 0$ and $\sup \sum_{j=0}^{\infty} x_j^2 = A^2$. So the possible values are $(0, A^2)$.

Example 0.2 (2000 A-4). Show that the improper integral

$$\lim_{B \to \infty} \int_0^B \sin(x) \sin(x^2) \, dx$$

converges.

Write

$$\int_0^B \sin(x) \sin(x^2) \, dx = \frac{1}{2} \int_0^B \cos(x^2 - x) \, dx - \frac{1}{2} \int_0^B \cos(x^2 + x) \, dx$$

Substitutions $u = x^2 \pm x$ will do the trick.

Example 0.3 (2000 B-3). Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi j t)$, where each $a_j$ is real and $a_N$ is not equal to 0. Let $N_k$ denote the number of zeroes (including multiplicities) of $\frac{df}{dt}$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \text{ and } \lim_{k \to \infty} N_k = 2N.$$[Editorial clarification: only zeroes in $[0,1)$ should be counted.]

The first statement follows directly from Rolle’s theorem (Mean Value Theorem). For the second part, note that

$$\lim_{k \to \infty} \frac{1}{(2\pi N)^k} f^{(k)}(x) = \pm \sin(2\pi N x) \text{ or } \pm \cos(2\pi N x)$$

You just have to show the following: if $f_n(x)$ converges uniformly to $f(x)$ as $n \to \infty$ on $[a, b]$, $f(x)$ has only finitely many zeroes on $[a, b]$ and $f(a)f(b) \neq 0$, then $f_n(x)$ and $f(x)$ has the same number zeroes as $n$ is large enough.

Example 0.4 (2000 B-4). Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all $x$. Show that $f(x) = 0$ for $-1 \leq x \leq 1$. 


If we let \( x = \cos(\theta) \), then the relation \( f(2x^2 - 1) = 2xf(x) \) becomes

\[
f(\cos(2\theta)) = 2 \cos(\theta)f(\cos(\theta))
\]

Use the formula \( \sin(2\theta) = 2\sin(\theta)\cos(\theta) \), we rewrite it as

\[
\frac{f(\cos(2\theta))}{\sin(2\theta)} = \frac{f(\cos(\theta))}{\sin(\theta)}
\]

Let \( g(\theta) = f(\cos(\theta))/\sin(\theta) \). So \( g(2\theta) = g(\theta) \) (continuity needs some verification). I will let you supply the rest of the argument for \( g(\theta) \equiv 0 \).

**Example 0.5** (1999 A-1). Find polynomials \( f(x), g(x), \) and \( h(x) \), if they exist, such that for all \( x \),

\[
|f(x)| - |g(x)| + h(x) = \begin{cases} 
-1 & \text{if } x < -1 \\
3x + 2 & \text{if } -1 \leq x \leq 0 \\
-2x + 2 & \text{if } x > 0.
\end{cases}
\]

Note that \( |f(x)| = af(x) \) for \( a = \pm 1 \). So

\[
\begin{cases} 
 a_1f(x) + a_2g(x) + h(x) = -1 \\
a_3f(x) + a_4g(x) + h(x) = 3x + 2 \\
a_5f(x) + a_6g(x) + h(x) = -2x + 2
\end{cases}
\]

where \( a_i = \pm 1 \). So

\[
\begin{cases} 
 b_1f(x) + b_2g(x) = -3 - 3x \\
b_3f(x) + b_4g(x) = 5x
\end{cases}
\]

where \( b_i = -2, 0, 2 \). So \( f(x) \) and \( g(x) \) are both of degree 1. One of them is \( ax \) and the other is \( b(x-1) \).

**Example 0.6** (1999 A-2). Let \( p(x) \) be a polynomial that is nonnegative for all real \( x \). Prove that for some \( k \), there are polynomials \( f_1(x), \ldots, f_k(x) \) such that

\[
p(x) = \sum_{j=1}^{k} (f_j(x))^2.
\]

Note that \( p(x) = \prod Q_i(x) \), where each \( Q_i(x) = ax^2 + bx + c \) is a quadratic polynomial with \( a > 0 \) and \( b^2 - 4ac \leq 0 \). We can write \( Q_i(x) = (g(x))^2 + (h(x))^2 \). Note that

\[
(g^2 + h^2)(m^2 + n^2) = (gm - hn)^2 + (gn + mh)^2
\]

**Example 0.7** (1999 A-5). Prove that there is a constant \( C \) such that, if \( p(x) \) is a polynomial of degree 1999, then

\[
|p(0)| \leq C \int_{-1}^{1} |p(x)| \, dx.
\]
We prove the following: Let $p(x)$ be a polynomial of degree $d$ and $p(0) = 1$. Then there exists a number $C_d > 0$ such that

$$\int_{-1}^{1} |p(x)|dx \geq C_d.$$

If all of roots of $f(x)$ are in the disc $|x| < 1/2$, then

$$\int_{-1}^{1} |p(x)|dx \geq (\int_{-1}^{-3/4} + \int_{3/4}^{1})|p(x)|dx > 2^{-d}.$$

If one of root of $f(x)$ is outside of the disc $|x| < 1/2$, we write $p(x) = q(x)(1 - \alpha x)$ with $|\alpha| \leq 2$. Then

$$\int_{-1}^{1} |p(x)|dx \geq \int_{-1/4}^{1/4} |q(x)||1 - \alpha x|dx$$

$$\geq \frac{1}{2} \int_{-1/4}^{1/4} |q(x)|dx$$

$$\geq \frac{1}{8} \int_{-1}^{1} |q(x/4)|dx \geq \frac{C_{d-1}}{8}$$

So it suffices to pick $C_d = \min(C_{d-1}/8, 1/2^d)$. 