CALCULUS PROBLEMS IN PUTNAM EXAM

Calculus problems are by far the most common type of problems in Putnam. The techniques involved are way too diversed to classify. I will use examples to introduce some important techniques.

Example 0.1 (2000 A-1). *Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?*

This is probably one of the easiest Putnam problem: $\inf \sum x_j^2 = 0$ and $\sup \sum x_j^2 = A^2$. So the possible values are $(0, A^2)$.

Example 0.2 (2000 A-4). *Show that the improper integral*

$$\lim_{B\to\infty} \int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

Write

$$\int_0^B \sin(x)\sin(x^2)dx = \frac{1}{2}\int_0^B \cos(x^2 - x)dx - \frac{1}{2}\int_0^B \cos(x^2 + x)dx$$

Substitutions $u = x^2 \pm x$ will do the trick.

Example 0.3 (2000 B-3). *Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeroes (including multiplicities) of $\frac{d^k f}{dt^k}$. Prove that*

$$N_0 \le N_1 \le N_2 \le \cdots \text{ and } \lim_{k\to\infty} N_k = 2N.$$

[Editorial clarification: only zeroes in $[0, 1)$ should be counted.]

The first statement follows directly from Rolle's theorem (Mean Value Theorem). For the second part, note that

$$\lim_{k\to\infty} \frac{1}{(2\pi N)^k} f^{(k)}(x) = \pm\sin(2\pi Nx) \text{ or } \pm\cos(2\pi Nx)$$

You just have to show the following: if $f_n(x)$ converges uniformly to $f(x)$ as $n \to \infty$ on $[a, b]$, $f(x)$ has only finitely many zeroes on $[a, b]$ and $f(a)f(b) \ne 0$, then $f_n(x)$ and $f(x)$ has the same number zeroes as n is large enough.

Example 0.4 (2000 B-4). *Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \le x \le 1$.*

If we let $x = \cos(\theta)$, then the relation $f(2x^2 - 1) = 2xf(x)$ becomes

$$f(\cos(2\theta)) = 2\cos(\theta)f(\cos(\theta))$$

Use the formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we rewrite it as

$$\frac{f(\cos(2\theta))}{\sin(2\theta)} = \frac{f(\cos(\theta))}{\sin(\theta)}$$

Let $g(\theta) = f(\cos(\theta))/\sin(\theta)$. So $g(\theta)$ is continuous and periodic and satisfies $g(2\theta) = g(\theta)$ (continuity needs some verification). I will let you supply the rest of the argument for $g(\theta) \equiv 0$.

Example 0.5 (1999 A-1)**.** *Find polynomials $f(x)$,$g(x)$, and $h(x)$, if they exist, such that for all x,*

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \le x \le 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

Note that $|f(x)| = af(x)$ for $a = \pm 1$. So

$$\begin{cases} a_1 f(x) + a_2 g(x) + h(x) = -1 \\ a_3 f(x) + a_4 g(x) + h(x) = 3x + 2 \\ a_5 f(x) + a_6 g(x) + h(x) = -2x + 2 \end{cases}$$

where $a_i = \pm 1$. So

$$\begin{cases} b_1 f(x) + b_2 g(x) = -3 - 3x \\ b_3 f(x) + b_4 g(x) = 5x \end{cases}$$

where $b_i = -2, 0, 2$. So $f(x)$ and $g(x)$ are both of degree 1. One of them is ax and the other is $b(x - 1)$.

Example 0.6 (1999 A-2)**.** *Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x), \dots, f_k(x)$ such that*

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

Note that $p(x) = \prod Q_i(x)$, where each $Q_i(x) = ax^2 + bx + c$ is a quadratic polynomial with $a > 0$ and $b^2 - 4ac \le 0$. We can write $Q_i(x) = (g(x))^2 + (h(x))^2$. Note that

$$(g^2 + h^2)(m^2 + n^2) = (gm - hn)^2 + (gn + mh)^2$$

Example 0.7 (1999 A-5)**.** *Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then*

$$|p(0)| \le C \int_{-1}^{1} |p(x)|\, dx.$$

We prove the following: Let $p(x)$ be a polynomial of degree d and $p(0) = 1$. Then there exists a number $C_d > 0$ such that

$$\int_{-1}^{1} |p(x)|dx \geq C_d.$$

If all of roots of $f(x)$ are in the disc $|x| < 1/2$, then

$$\int_{-1}^{1} |p(x)|dx \geq (\int_{-1}^{-3/4} + \int_{3/4}^{1})|p(x)|dx > 2^{-d}.$$

If one of root of $f(x)$ is outside of the disc $|x| < 1/2$, we write $p(x) = q(x)(1 - \alpha x)$ with $|\alpha| \leq 2$. Then

$$\begin{aligned}\int_{-1}^{1} |p(x)|dx &\geq \int_{-1/4}^{1/4} |q(x)||1 - \alpha x|dx \\ &\geq \frac{1}{2}\int_{-1/4}^{1/4} |q(x)|dx \\ &\geq \frac{1}{8}\int_{-1}^{1} |q(x/4)|dx \geq \frac{C_{d-1}}{8}\end{aligned}$$

So it suffices to pick $C_d = \min(C_{d-1}/8, 1/2^d)$.