GENERATING FUNCTION

1. Examples

Question 1.1. *Toss a coin n times and find the probability of getting exactly k heads.*

Represent H by x^1 and T by x^0 and a sequence, say, "HTHHT" by $(x^1)(x^0)(x^1)(x^1)(x^0)$. We see that all the possible outcomes of n tosses are represented by the expansion of $f(x) = (x^0+x^1)^n = (1+x)^n$. The coefficient of x^k in $f(x)$ is the number of outcomes with exactly k heads. Assuming the coin is fair, we see that the probability is $\binom{n}{k}/2^n$.

Next, let us factor the probability into the generating function $f(x)$. By that we mean we want the coefficient of x^k gives the probability of getting exactly k heads. To make the problem less trivial, let us assume that the coin is not necessarily fair with p for head and q for tail ($p + q = 1$). We want to find $f(x) = \sum a_k x^k$ with a_k the probability of getting exactly k heads. It is not very hard to come up with

$$f(x) = (q + px)^n = \sum a_k x^k$$

So $a_k = p^k q^{n-k}\binom{n}{k}$. A more complicated version of the problem is the following.

Question 1.2. *You have coins $C_1, ..., C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k + 1)$ of falling heads. If the n coins are tossed, what is the probability of an odd number of heads?*

Let $p_k = 1/(2k+1)$ and $q_k = 1-p_k$. It is not hard to set up the generating function as in the previous example

$$f(x) = (q_1 + p_1x)(q_2 + p_2x)...(q_n + p_nx^n) = \sum a_m x^m$$

with a_m the probability of getting exactly m heads. We are looking for $a_1 + a_3 + a_5 + ...$

Here is a neat trick to find $a_1+a_3+a_5+...$ given a series $f(x) = \sum a_m x^m$. Note that

$$f(1) = \sum a_m = \sum a_{2k} + \sum a_{2k+1}$$

and

$$f(-1) = \sum a_m(-1)^m = \sum a_{2k} - \sum a_{2k+1}$$

So

$$\sum a_{2k+1} = \frac{1}{2}(f(1) - f(-1)).$$

In our case, $f(1) = 1$ and $f(-1) = 1/(2n+1)$. So the probabily of getting odd number of heads is $n/(2n + 1)$.

- Given $f(x) = \sum a_m x^m$. What if we want to find $\sum a_0 + a_3 + a_6 + ... = \sum a_{3k}$?
- More generally, What if we want to find $\sum a_{pk+r}$ for some fixed p and r. Hint: use roots of unity.

Question 1.3. *Find the number of ways of changing a 500 dollar bill into \$1, \$2, \$5, \$10, \$20's.*

Let k_1, k_2, k_3, k_4, k_5 be the number of \$1, \$2, \$5, \$10, \$20's, respectively. We are looking at the numbers of nonnegative integral solutions of the equation:

$$k_1 + 2k_2 + 5k_3 + 10k_4 + 20k_5 = 500.$$

Set up the correspondence

$$(k_1, k_2, k_3, k_4, k_5) \to (x^{k_1})(x^{2k_2})(x^{5k_3})(x^{10k_4})(x^{20k_5})$$

Then the number of the solutions of the equation is the coefficient of x^{500} in

$$\begin{aligned} f(x) &= \left(\prod_{k_1=0}^{\infty} x^{k_1}\right)\left(\prod_{k_2=0}^{\infty} x^{2k_2}\right)\left(\prod_{k_3=0}^{\infty} x^{5k_3}\right)\left(\prod_{k_4=0}^{\infty} x^{10k_4}\right)\left(\prod_{k_5=0}^{\infty} x^{20k_5}\right) \\ &= (1 + x + x^2 + ...)(1 + x^2 + x^4 + ...)(1 + x^5 + x^{10} + ...) \\ &\quad (1 + x^{10} + x^{20} + ...)(1 + x^{20} + x^{40} + ...) \\ &= \frac{1}{(1-x)(1-x^2)(1-x^5)(1-x^{10})(1-x^{20})} \end{aligned}$$

Of course, to find the Taylor expansion of $f(x)$, we have to write $f(x)$ as a sum of partial fractions, which is doable in theory but impossible by hand. But we can still say a lot about the coefficients of $f(x)$. Let $f(x) = \sum p(n)x^n$, i.e., $p(n)$ is the number of the ways changing n dollar. Then the following is true:

- $p(n)$ grows at the order of n^4 as $n \to \infty$. I leave this as an exercise.
- $p(20n)$ is a polynomial in n of degree 4. This is a little bit harder to prove.
- $$p(20n) = \sum_{k=0}^{4} p(20k) \prod_{\substack{0 \le l \le 4 \\ l \ne k}} \frac{n-l}{k-l}$$

  This uses interpolation formula. Once we know that $p(20n)$ is a polynomial of degree 4 and the values of $p(20n)$ at five points, say $p(0), p(20), p(40), p(60), p(80)$, we can find $p(20n)$ by interpolation.

  More generally, the number of nonnegative integral solutions $(x_1, x_2, ..., x_m)$ of the equation

  $$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_m x_m = n$$

is given by the coefficients of the generating function

$$f(x) = \frac{1}{(1-x^{\lambda_1})(1-x^{\lambda_2})...(1-x^{\lambda_m})}.$$

- What if we are looking for integral solutions x_i within the range, say $\alpha_i \le x_i \le \beta_i$? What is the corresponding generating function?
- What if we are looking for odd solutions, solutions with $x_i \equiv 1 (\bmod 3)$ and etc? What are the corresponding generating functions?

Question 1.4. *Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$?*

Let $P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_k x^k + ...$ We are looking at the solutions of

$$a_0 + 2a_1 + 4a_2 + ... + 2^k a_k + ... = n$$

with $0 \le a_k \le 3$. The number of such solutions are given by the coefficient of x^n in

$$\begin{aligned} f(x) &= (1+x+x^2+x^3)(1+x^2+x^4+x^6)(1+x^4+x^8+x^{12})... \\ &= \prod_{k=0}^{\infty}(1+x^{2^k}+x^{2(2^k)}+x^{3(2^k)}) \\ &= \prod_{k=0}^{\infty}\frac{1-x^{2^{k+2}}}{1-x^{2^k}} = \frac{1}{(1-x)(1-x^2)} \\ &= \left(\frac{1}{4}\right)\frac{1}{1-x} + \left(\frac{1}{2}\right)\frac{1}{(1-x)^2} + \left(\frac{1}{4}\right)\frac{1}{1+x} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{4}+\frac{1}{4}(-1)^n+\frac{1}{2}(n+1)\right)x^n \end{aligned}$$

Another way to express the result is $\lfloor n/2 \rfloor + 1$.

- Find a purely combinatorial solution to the problem.

Question 1.5. *Find the number of subsets of $\{1, ..., 2003\}$, the sum of whose elements is divisible by 5.*

Set up the correspondence:

$$S = \{s_1, s_2, ..., s_k\} \to \sigma(S) = x^{s_1}x^{s_2}...x^{s_k}.$$

Then

$$\sum_{S\subset\{1,...,2003\}} \sigma(S) = f(x) = (1+x)(1+x^2)...(1+x^{2003})$$

The coefficient of x^n in $f(x)$ is the number of subsets of $\{1, ..., 2003\}$, the number of whose elements is exactly n. So if $f(x) = \sum a_n x^n$, we are looking for the sum $a_0 + a_5 + a_{10} + ...$ Let $r_n = \exp(2n\pi i/5)$ be the 5-th roots of unity. Then

$$a_0 + a_5 + a_{10} + ... = \frac{1}{5}(f(r_0) + f(r_1) + f(r_2) + f(r_3) + f(r_4)).$$

Obviously, $f(r_0) = 2^{2003}$. For $1 \leq k \leq 4$,

$$\begin{aligned} f(r_k) &= \left((1+r_k^0)(1+r_k^1)(1+r_k^2)(1+r_k^3)(1+r_k^4)\right)^{400} \\ &\quad (1+r_k^1)(1+r_k^2)(1+r_k^3) \\ &= 2^{400}(2 + 2r_k + r_k^2 + 2r_k^3 + r_k^4) \end{aligned}$$

and hence

$$\sum_{k=1}^{4} f(r_k) = 2^{400}(8 - 2 - 1 - 2 - 1) = 2^{401}.$$

So the answer is $(1/5)(2^{2003} + 2^{401})$.

- Replace 2003 by $n = 5k + l$ and redo the computation.
- Find a purely combinatorial solution to the problem.

Question 1.6. *Suppose that each of n people writes down the numbers $1, 2, 3$ in random order in one column of a $3 \times n$ matrix, with all orders being equally likely and independent. Show that for some $n \geq 1995$, the event that the row sums are consecutive integers is at least four times likely as the event that the row sums are equal.*

Instead of $(1, 2, 3)$, we use $(-1, 0, 1)$. Set up the correspondence:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to x^a y^b$$

Then a $3 \times n$ such matrix can be represented by a term in the expansion of

$$\phi_n(x, y) = \left(x + y + \frac{x}{y} + \frac{y}{x} + \frac{1}{x} + \frac{1}{y}\right)^n$$

Let a_n be the constant term in $\phi_n(x, y)$. It is not hard to see that a_n is exactly the number of matrices with equal row sums. Let $b_n, c_n, d_n, e_n, f_n, g_n$ be the coefficients of $x, y, x/y, y/x, 1/x, 1/y$ in $\phi_n(x, y)$. It is not hard to see that $b_n + c_n + d_n + e_n + f_n + g_n$ are the number of matrices with consecutive row sums. Actually, from

$$\phi_{n+1}(x, y) = \left(x + y + \frac{x}{y} + \frac{y}{x} + \frac{1}{x} + \frac{1}{y}\right)\phi_n(x, y)$$

we see that

$$a_{n+1} = b_n + c_n + d_n + e_n + f_n + g_n$$

We are essentially asked to show that $a_n/a_{n+1} \leq 1/4$ for n large. In other word, if we put $\varphi(t) = \sum a_n t^n$, it suffices to show that the radius of convergence of $\varphi(t)$ is less than $1/4$. This suggests us to put all $\phi_n(x,y)$ together into

$$H(x,y,t) = \sum \phi_n(x,y)t^n$$

If we expand $H(x,y,t)$ as an infinite series in (x,y) (one technical issue here: $H(x,y,t)$ expands as a Laurent series), $\varphi(t)$ is the constant term.

The rest of the proof is quite technical. You need to know some complex analysis to understand it. I will list the key steps.

First, compute $H(x,y,t)$:

$$\begin{aligned} H(x,y,t) &= \frac{xy}{xy - t(x^2y + xy^2 + x^2 + y^2 + x + y)} \\ &= \frac{xy}{-t(y+y^2) + (y - ty^2 - t)x - (ty+t)x^2} \end{aligned}$$

Second, expand $H(x,y,t)$ in x gives us that $\varphi(t)$ is the constant term in the Laurent series of

$$\frac{y}{\sqrt{(y - ty^2 - t)^2 - 4t^2(y+1)^2y}}$$

Then

$$\varphi(t) = \frac{1}{2\pi i}\int_C \frac{y}{\sqrt{(y - ty^2 - t)^2 - 4t^2(y+1)^2y}}dy$$

where $C = \{|y| = 1\}$ is the unit circle.

Finally, it takes some effort to show that $\varphi(t)$ is a meromorphic function in t and it has singularities when $\Delta(y) = (y - ty^2 - t)^2 - 4t^2(y+1)^2y$ has a double root on the unit circle. This gives us singularities of $\varphi(t)$ at $t = 1/6$ and $t = -1/2$. So the radius of convergence of $\varphi(t)$ is at most $1/6$.

2. Exercise

Question 2.1. *Let $p(n)$ be the number of ways of changing n dollars into coins and bills of \$1, \$2, \$5, \$10, \$20's. Show that*

$$\lim_{n\to\infty} \frac{p(n)}{n^4} = \frac{1}{2\cdot 5\cdot 10\cdot 20}.$$

Question 2.2. *Throw a dice n times and find the probability of getting a sum which is divisible by 5.*