# ALGEBRA AND NUMBER THEORY

## 1. Linear Algebra

**Question 1.1.** *Are the following matrices invertible?*

(1)
$$\begin{pmatrix} 1235 & 2344 & 1234 & 1990 \\ 2124 & 4123 & 1990 & 3026 \\ 1230 & 1234 & 9095 & 1230 \\ 1262 & 2312 & 2324 & 3907 \end{pmatrix}$$

(2)
$$\begin{pmatrix} 2003 & 1 & 1 & ... & 1 \\ 1 & 2003 & 1 & ... & 1 \\ 1 & 1 & 2003 & ... & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & ... & 2003 \end{pmatrix}_{2003\times 2003}$$

(3) $(a_{ij})_{n\times n}$ *with* $a_{ii} = \pi$ *and* $a_{ij} \in \mathbb{Q}$ *for* $i \neq j$.

**Question 1.2.** *Let* $A$ *and* $B$ *be* $2 \times 2$ *matrices with integer entries such that each of* $A$, $A + B$, $A + 2B$, $A + 3B$, $A + 4B$ *has an inverse with integer entries. Show that the same is true for* $A + 5B$.

## 2. Abstract Algebra

**Question 2.1.** *Prove or disprove: Two groups are isomorphic if each of them is isomorphic to a subgroup of the other.*

This is obviously true if the groups are finite. Consider two infinite groups which are not isomorphic, e.g., $\mathbb{Z}$ and $\mathbb{Q}$. They are not isomorphic because $\mathbb{Q}$ is divisible and $\mathbb{Z}$ is not (for every positive integer $n$ and every $x \in \mathbb{Q}$, there exists $y \in \mathbb{Q}$ such that $x = ny$; the same is obviously false for $\mathbb{Z}$.) $\mathbb{Z}$ can be embedded into $\mathbb{Q}$ but $\mathbb{Q}$ cannot be embedded into $\mathbb{Z}$. So we replace $\mathbb{Z}$ by $\mathbb{Z} \times \mathbb{Q}$. Now we can embed $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{Q}$ but cannot embed $\mathbb{Z} \times \mathbb{Q}$ into $\mathbb{Q}$. So we replace $\mathbb{Q}$ by $\mathbb{Q} \times \mathbb{Q}$. Eventually, we will arrive at

$$A = \mathbb{Z} \times \prod_{n=1}^{\infty} \mathbb{Q} = \{(x_0, x_1, x_2, ...) : x_0 \in \mathbb{Z}, x_n \in \mathbb{Q} \text{ for } n > 0\}$$

and

$$B = \prod_{n=1}^{\infty} \mathbb{Q} = \{(x_1, x_2, ...) : x_n \in \mathbb{Q}\}$$

The embeddings $A \hookrightarrow B$ and $B \hookrightarrow A$ are the obvious ones but $A$ and $B$ are not isomorphic because $B$ is divisible while $A$ is not.