1. Problems

We will discuss the following problems on Oct. 20 & 27, 2011.

Question 1.1. *Prove that* $p_n > 2n$ *for every* $n \geq 5$*, where* p_n *denote the* n*-th prime number* ($p_1 = 2$).

Question 1.2. *There are given five line segments having the property that with any three of them can be the sides of a triangle. Prove that at least one of these triangles must be acute.*

Question 1.3. *Let* A *be a positive real number. What are the possible values of* $\sum_{j=0}^{\infty} x_j^2$*, given that* $x_0, x_1, \ldots$ *are positive numbers for which* $\sum_{j=0}^{\infty} x_j = A$*?*

Question 1.4. *Let* a, b, c *be positive numbers such that* $abc = 18$*. Prove that*

$$\frac{a^3 + b^3 + c^3}{3} \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

Question 1.5. *Let* $x_1, x_2, ..., x_{2010}$ *be positive real numbers such that*

$$\frac{1}{2009 + x_1} + \frac{1}{2009 + x_2} + ... + \frac{1}{2009 + x_{2010}} > 1.$$

Prove that

$$x_1 x_2 ... x_{2010} < 1.$$

Question 1.6. *Find all functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$f(x + \sqrt{x}) \leq x \leq f(x) + \sqrt{f(x)}$$

for all $x \in \mathbb{R}^+$.

Question 1.7. *Show that the improper integral*

$$\lim_{B \to \infty} \int_0^B \sin(x)\sin(x^2)\, dx$$

converges.

Question 1.8. *It is well known that the harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ...$$

diverge.

(1) *If we drop a term* $1/n$ *if the decimal representation of* n *contains* 9*, does the resulting series converge or diverge?*
(2) *Similarly, if we drop a term* $1/n$ *if the decimal representation of* n *contains two consecutive 9s, does the resulting series converge or diverge? For example, we drop* $1/99$, $1/199$, $1/995$, *but not* $1/9$ *or* $1/919$.