UNIRATIONALITY OF FANO VARIETIES

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1. Introduction

This note is an extension of the paper [HMP]. The main theorem of [HMP] states that a relatively smooth hypersurface (i.e. a hypersurface whose singular locus has sufficient large codimension with respect to its degree) is unirational. A mild modification can generalize this to complete intersections. As an application, we will show the Fano variety of a relatively smooth hypersurface is also unirational. Precisely, we have

Theorem 1.1 (Uniratonality of Fano Varieties). *Let X be a degree d hypersurface over an algebraically closed field. The Fano variety $F_k(X)$ of X is unirational if the codimension of singular locus of X is sufficiently large with respect to both d and k. Here by "sufficiently large", we mean for any integers d and k, there exists a number $N(d,k)$ such that $F_k(X)$ is unirational if the codimension of singular locus of X is greater than $N(d,k)$.*

Remarks.

(1) Throughout the note, we will work exclusively on the fields of characteristics 0.
(2) The Fano variety $F_k(X)$ of a projective variety $X \subset \mathbb{P}^n$ is defined to be the subvariety of the grassmannian $\mathbb{G}(k,n)$ consisting of k-planes contained in X.

Theorem 1.1 will be proved in the next section and followed by the discussion of some of its implications. As we will see, in the course of the proof we will obtain the results of [HMP] on Fano varieties via a different approach.

I would like to thank J. Harris for suggesting this problem and proofreading the paper. Moreover, I have benefited greatly from the discussions with him.

It has come to my attention that these results were also obtained independently by Debarre and Manivel.

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2. Unirationality of Fano Varieties

With the setup in Theorem 1.1, we have X a hypersurface of degree d in $\mathbb{P}^n$ and $F_k(X)$ its Fano variety. Let $M(k+1,n+1)$ be the linear space of $(k+1)\times(n+1)$ matrices and $M_k(k+1,n+1)\subset M(k+1,n+1)$ be the determinantal variety of $(k+1)\times(n+1)$ matrices of rank $\le k$. There is a natural smooth surjective morphism

$$\varphi : \mathbb{P}M(k+1,n+1)\backslash M_k(k+1,n+1) \to \mathbb{G}(k,n).$$

Take $W=\varphi^{-1}(F_k(X))$. It is not hard to see

Proposition 2.1. *W is cut out by $\binom{d+k}{k}$ degree d hypersurfaces in $\mathbb{P}M(k+1,n+1)\backslash M_k(k+1,n+1)$.*

Proof. The proof is straightforward by explicitly writing down the defining equations of W in the following way.

Fix $r=\binom{d+k}{k}$ points $p_1,...,p_r\in\mathbb{P}^k$ in general position which impose independent conditions on $H^0(\mathbb{P}^k,\mathcal{O}(d))$. Let Λ be a k-plane in $\mathbb{P}^n$ represented by the $(k+1)\times(n+1)$ matrix

$$Y=\begin{pmatrix} Y_{00} & Y_{01} & \dots & Y_{0n}\\ Y_{10} & Y_{11} & \dots & Y_{1n}\\ \vdots & \vdots & \ddots & \vdots\\ Y_{k0} & Y_{k1} & \dots & Y_{kn}\end{pmatrix}. \tag{2.1}$$

Let $f:\mathbb{P}^k\to\Lambda$ be the linear map which sends

$$\begin{aligned}(1,0,0,...,0)&\to(Y_{00},Y_{01},...,Y_{0n}),\\ (0,1,0,...,0)&\to(Y_{10},Y_{11},...,Y_{1n}),\\ &...,\\ (0,0,...,0,1)&\to(Y_{k0},Y_{k1},...,Y_{kn}).\end{aligned}$$

Obviously, X contains Λ if and only if X passes through $f(p_1),f(p_2),...,f(p_r)$. Specifically, let X be defined by $G(X_0,X_1,...,X_n)=0$, $p_i=(u_{i0},u_{i1},...,u_{ik})$ for $i=1,2,...,r$, and $\mathcal{W}$ be the variety in $\mathbb{P}M(k+1,n+1)\cong\mathbb{P}^{(n+1)(k+1)-1}$ cut out by r degree d hypersurfaces

$$G_i(Y)=G\left(\sum_{j=0}^k u_{ij}Y_{j0},\sum_{j=0}^k u_{ij}Y_{j1},...,\sum_{j=0}^k u_{ij}Y_{jn}\right)=0 \tag{2.2}$$

for $i=1,2,...,r$. Our previous argument shows that

$$\mathcal{W}\cap(\mathbb{P}M(k+1,n+1)\backslash M_k(k+1,n+1))=W.$$

□

Let $\Phi \subset \mathcal{W}$ be the locus of the points at which the r degree d hypersurfaces defined by (2.2) fail to intersect transversely, i.e., the points where the Jacobian

(2.3)

$$\begin{pmatrix} u_{10}\frac{\partial G}{\partial X_0}(P_1) & \dots & u_{1k}\frac{\partial G}{\partial X_0}(P_1) & \dots & u_{10}\frac{\partial G}{\partial X_n}(P_1) & \dots & u_{1k}\frac{\partial G}{\partial X_n}(P_1) \\ u_{20}\frac{\partial G}{\partial X_0}(P_2) & \dots & u_{2k}\frac{\partial G}{\partial X_0}(P_2) & \dots & u_{20}\frac{\partial G}{\partial X_n}(P_2) & \dots & u_{2k}\frac{\partial G}{\partial X_n}(P_2) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ u_{r0}\frac{\partial G}{\partial X_0}(P_r) & \dots & u_{rk}\frac{\partial G}{\partial X_0}(P_r) & \dots & u_{r0}\frac{\partial G}{\partial X_n}(P_r) & \dots & u_{rk}\frac{\partial G}{\partial X_n}(P_r) \end{pmatrix}$$

has rank less than r, where

$$P_i = \left(\sum_{j=0}^{k} u_{ij}Y_{j0}, \sum_{j=0}^{k} u_{ij}Y_{j1}, ..., \sum_{j=0}^{k} u_{ij}Y_{jn}\right) \text{ for } i = 1, 2, ..., r.$$

Geometrically, Φ consists of the singular locus of $\mathcal{W}$ and irreducible components of $\mathcal{W}$ whose codimensions are less than r. Let $\Psi = \varphi(\Phi \cap W)$. Then Ψ consists of the singular locus of $F_k(X)$ and irreducible components of $F_k(X)$ whose dimensions are greater than the expected dimension.

Multiplying the Jacobian matrix (2.3) on the right by the $((k+1)(n+1)) \times (n+1)^2$ matrix

$$\begin{pmatrix} Y & & & \\ & Y & & \\ & & \ddots & \\ & & & Y \end{pmatrix}$$

where Y is the matrix given by (2.1), we have

$$\begin{pmatrix} G_{00}(P_1) & \dots & G_{n0}(P_1) & \dots & G_{0n}(P_1) & \dots & G_{nn}(P_1) \\ G_{00}(P_2) & \dots & G_{n0}(P_2) & \dots & G_{0n}(P_2) & \dots & G_{nn}(P_2) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ G_{00}(P_r) & \dots & G_{n0}(P_r) & \dots & G_{0n}(P_r) & \dots & G_{nn}(P_r) \end{pmatrix},$$

where $G_{ij}(X_0, X_1, ..., X_n) = X_i \dfrac{\partial G}{\partial X_j}$. Hence

> the Jacobian matrix (2.3) has rank r if and only if $H^0(\Lambda, \mathcal{O}(d))$ is spanned by $\{X_i \frac{\partial G}{\partial X_j}\Big|_{\Lambda} : 0 \leq i, j \leq n\}$. $(*)$

Our main theorem is

Theorem 2.1. *For any integers k and d, there exists a number $N = N(d, k)$ such that*

(1) *Let*
$$V(d,k) = \max\left\{\binom{k+d}{k}, N(d,k)\right\}.$$
Then $\operatorname{codim} X_{sing} \le V(d-1, \operatorname{codim}\Psi + k)$. *(We take* $N(0,k) = 0$.*)*

(2) *If X is any degree d hypersurface in projective space $\mathbb{P}^n$ and X_{sing} has codimension at least $N(d,k)$, then*
$$\dim F_k(X) = \phi(d,n,k),$$
where
$$\phi(d,n,k) = (k+1)(n-k) - \binom{d+k}{k}$$
is the expected dimension of $F_k(X)$.

(3) *For any $n \ge N(d,k)$ and any degree d hypersurface $X \subset \mathbb{P}^n$,*
$$\dim F_k(X) \le \max\{\phi(d,n,k), \phi(d,n,k) + \dim X_{sing} - 1\}.$$

(4) *Let $\{D_\lambda\}_{\lambda\in\mathbb{P}^m}$ be a base-point-free linear series of degree d hypersurfaces in $\mathbb{P}^n$ for $n \ge N(d,k)$. The the Fano correspondence $\{(\lambda,\Gamma) : \Gamma \subset D_\lambda\} \subset \mathbb{P}^m \times \mathbb{G}(k,n)$ has dimension $m + \phi(d,n,k)$ and each of its irreducible component dominates $\mathbb{P}^m$.*

(5) *If $n \ge N(d,k)$, a general k-plane $\Gamma \subset \mathbb{P}^n$ imposes independent conditions on any base-point-free linear series of degree d hypersurfaces in $\mathbb{P}^n$.*

Part (2)-(5) simply recover [HMP, Theorem 3.1], [HMP, Corollary 3.2] and [HMP, Corollary 3.3]. The key to the proof is part (1), which measures the dimension of the singular locus of $F_k(X)$.

First, we need to recall some facts about Schubert calculus on grassmannians.

Our notation will follow that of [G-H] except that $G(k,n)$ there should be read as $\mathbb{G}(k-1,n-1)$ here. Let $V_0 \subset V_1 \subset ... \subset V_{n-1} \subset \mathbb{P}^n$ be a flag of $\mathbb{P}^n$, i.e., V_i is an i-plane of $\mathbb{P}^n$. Let $n-k \ge a_0 \ge a_1 \ge ... \ge a_k$ be a sequence of integers. A Schubert class associated to $(a_0, a_1, ..., a_k)$ is an element in the cohomology ring (or Chow ring) of $\mathbb{G}(k,n)$ represented by the subvariety
$$\sigma_{a_0,a_1,\dots,a_k} = \{\Lambda \in \mathbb{G}(k,n) : \dim(\Lambda \cap V_{n-k+i-a_i}) \ge i, \forall 0 \le i \le k\}.$$

It is well known that the m-th integral cohomology (or m-th graded part of Chow ring) of $\mathbb{G}(k,n)$ is freely generated by Schubert classes $\sigma_{a_0,a_1,\dots,a_k}$ with $\sum a_i = m$ for $m = 0, 1, ..., 2n$ (see, for example, [G-H, Chap. 1, Sec. 5]).

Lemma 2.1. *Let $n \geq l > k$. Then for any Schubert class $\sigma_{b_0,b_1,\ldots,b_k}$ in $\mathbb{G}(k,n)$ with $\sum b_i \leq l-k$,*

$$\sigma_{n-l,n-l,\ldots,n-l} \cdot \sigma_{b_0,b_1,\ldots,b_k} \neq 0,$$

which implies that any subvariety of $\mathbb{G}(k,n)$ with codimension $l-k$ or less must have nonempty intersection with any cycle of type $\sigma_{n-l,n-l,\ldots,n-l}$.

Proof. By the reduction formula for Schubert classes [G-H, p. 202],

$$\begin{aligned} & \sigma_{n-l,n-l,\ldots,n-l} \cdot \sigma_{b_0,b_1,b_2,\ldots,b_{k-1},b_k} \cdot \sigma_{l-k-b_k,l-k-b_{k-1},\ldots,l-k-b_2,l-k-b_1,l-k-b_0} \\ = \ & (\sigma_{n-l,n-l,\ldots,n-l} \cdot \sigma_{b_1,b_2,\ldots,b_{k-1},b_k} \cdot \sigma_{l-k-b_k,l-k-b_{k-1},\ldots,l-k-b_2,l-k-b_1})_{\mathbb{G}(k-1,n-1)} \\ = \ & (\sigma_{n-l,n-l,\ldots,n-l} \cdot \sigma_{b_2,\ldots,b_{k-1},b_k} \cdot \sigma_{l-k-b_k,l-k-b_{k-1},\ldots,l-k-b_2})_{\mathbb{G}(k-2,n-2)} \\ & \ldots \\ = \ & 1. \end{aligned}$$

□

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We will prove it by induction on d. If $d = 1$, we may simply take $N(1,k) = 0$.

Let us first show part (1). The rest will follow easily. Let l be the number satisfying $V(d-1,l) < \operatorname{codim} X_{sing} \leq V(d-1,l+1)$. If $l \leq k$, it follows immediately that

$$\operatorname{codim} X_{sing} \leq V(d-1,l+1) \leq V(d-1,\operatorname{codim}\Psi + k).$$

So we may assume $l > k$.

Since $\operatorname{codim} X_{sing} > V(d-1,l)$, there exists a $V(d-1,l)$-plane Θ disjoint from the singular locus of X and hence the linear series spanned by $\{\frac{\partial G}{\partial X_i} : 0 \leq i \leq n\}$ is base point free when restricted to Θ, where $G(X_0, X_1, ..., X_n) = 0$ is the defining equation of X as used before.

By the induction hypothesis of (5), a general l-plane in $\mathbb{P}^n$ imposes independent conditions on any base point free linear series of degree $d-1$ hypersurfaces in $\mathbb{P}^n$ if $n \geq N(d-1,l)$. And since $\dim\Theta = V(d-1,l) \geq N(d-1,l)$, there exists an l-plane $\Gamma \subset \Theta$ imposing independent conditions on the linear series spanned by $\{\frac{\partial G}{\partial X_i}\Big|_\Theta : 0 \leq i \leq n\}$. And since

$$n \geq \dim\Theta \geq \binom{d-1+l}{l},$$

$\{\frac{\partial G}{\partial X_i}\Big|_\Gamma : 0 \leq i \leq n\}$ actually spans $H^0(\Gamma, \mathcal{O}(d-1))$. Consequently, $\{G_{ij}\big|_\Gamma : 0 \leq i,j \leq n\}$ spans $H^0(\Gamma, \mathcal{O}(d))$, where $G_{ij}(X_0, X_1, ..., X_n) = X_i \frac{\partial G}{\partial X_j}$. Hence $\{G_{ij}\big|_\Lambda : 0 \leq i,j \leq n\}$ also spans $H^0(\Lambda, \mathcal{O}(d))$ for any k-plane Λ contained in Γ.

So by our previous observation (∗), the r degree d hypersurfaces given by (2.2) intersect transversely at any k-plane contained in Γ, i.e.,

$$\{\Lambda \in \mathbb{G}(k,n) : \Lambda \subset \Gamma\} \cap \Psi = \emptyset.$$

Since the variety $\{\Lambda \in \mathbb{G}(k,n) : \Lambda \subset \Gamma\}$ has type $\sigma_{n-l,n-l,\ldots,n-l}$, $\operatorname{codim}\Psi \geq l+1-k$ by Lemma 2.1. Hence

$$\operatorname{codim} X_{sing} \leq V(d-1, l+1) \leq V(d-1, \operatorname{codim}\Psi + k).$$

(1) ⇒ (2). Take

$$N = V\left(d-1, \binom{d+k}{k} + k - 1\right) + 1.$$

If $\operatorname{codim} X_{sing} \geq N$, by part (1), we must have $\operatorname{codim}\Psi \geq \binom{d+k}{k}$. Hence $F_k(X)$ has the expected dimension.

(2) ⇒ (3) ⇒ (4) ⇒ (5) follows the arguments in [HMP, Sec. 3]. □

If $F_k(X)$ has the expected dimension, W is the complete intersection of the r degree d hypersurfaces given by (2.2) in $\mathbb{P}M(k+1,n+1)\backslash M_k(k+1,n+1)$. If we further let $n-k+1 > r$, since $\mathbb{P}M_k(k+1,n+1)$ has codimension $n-k+1$ in $\mathbb{P}M(k+1,n+1)$, $\mathcal{W}$ agrees with $\overline{W}$, the closure of W in $\mathbb{P}M(k+1,n+1)$, and consequently $\overline{W}$ is the complete intersection of the r degree d hypersurfaces. And in this case

$$\Phi = \overline{W}_{sing} = (\mathbb{P}M_k(k+1,n+1) \cap \overline{W}) \cup \varphi^{-1}(\Psi).$$

Therefore

$$\operatorname{codim} \overline{W}_{sing} \geq \min\{n-k+1, \operatorname{codim}\Psi\}.$$

In summary, we have

Corollary 2.1. *When the codimension of the singular locus of X is sufficiently large with respect to both d and k, $\overline{W}$ is the complete intersection of $\binom{d+k}{k}$ degree d hypersurfaces in $\mathbb{P}M(k+1,n+1) \cong \mathbb{P}^{(n+1)(k+1)-1}$, and its singular locus consists of the boundary $\mathbb{P}M_k(k+1,n+1) \cap \overline{W}$ and $\varphi^{-1}(\Psi)$. Hence* $\operatorname{codim} \overline{W}_{sing} \to \infty$ *when* $\operatorname{codim} X_{sing} \to \infty$ *with d and k fixed.*

Then Theorem 1.1 follows immediately from [HMP, Theorem 4.1] and Corollary 2.1. Also notice that if X contains an l-plane Γ and we let

$$\Omega = \{\Lambda \in \mathbb{G}(k,n) : \Lambda \subset \Gamma\},$$

then $\varphi^{-1}(\Omega)$ is a plane in $\mathbb{P}M(k+1,n+1)$ of dimension $(l+1)(k+1)-1$. Furthermore, if Γ represents a nonsingular point of $F_l(X)$, $\varphi^{-1}(\Omega)$ is contained in the smooth locus of W. It is not hard to see from the proof of Theorem 2.1 that for any number l there exists a number $l' \geq l$ such that if Γ' is an l'-plane contained in X and disjoint from the singular locus of X, Γ' must contain at least one l-plane which represents a nonsingular point of $F_l(X)$. In summary, we have a "relative" version for Theorem 1.1 over number fields as the same for [HMP, Theorem 4.1].

Theorem 2.2. *Let X be a degree d hypersurface. Then for any integers d and k, there exists a number $l(d,k)$ such that the Fano variety $F_k(X)$ of X is unirational if X contains a plane which is disjoint from the singular locus of X and has dimension greater than $l(d,k)$.*

One consequence of Theorem 1.1 is that the generic fiber of the unirational morphism constructed in [HMP, Theorem 4.1] is geometrically unirational. Let us recall the construction in [HMP].

Let X be a hypersurface of degree d in $\mathbb{P}^n$ and $\Lambda \cong \mathbb{P}^m$ an m-plane contained in X and disjoint from the singular locus of X. Let

$$0 = m_1 < m_2 < ... < m_{d-1} = m$$

be an increasing sequence of integers, starting from 0 and running up to m, and let

$$F = F_{m_1,m_2,...,m_{d-1}}(\Lambda)$$

be the variety of flags

$$\Lambda_1 \subset \Lambda_2 \subset ... \subset \Lambda_{d-1} = \Lambda$$

in Λ. Let

$$G = F_{m_1+1,m_2+1,...,m_{d-1}+1}(\mathbb{P}^n)$$

be the variety of flags in the ambient projective space, and finally let

$$H \subset F \times G$$

be the closure of the locus of pairs $(\Lambda_1, \Lambda_2, ..., \Lambda_{d-1}; \Gamma_1, \Gamma_2, ..., \Gamma_{d-1})$ such that

$$\text{mult}_{\Lambda_i}(X \cap \Lambda_{i+1}) = d - i$$
$$\Lambda_i \text{ is a general } m_i\text{-plane in } \Lambda_{i+1}$$

and

$$\Gamma_i \text{ is a general } m_i\text{-plane containing } \Lambda_i \text{ and contained in } \Lambda_{i+1}$$

for $i = 1, ..., d-1$.

It is shown in [HMP, Theorem 4.1] that $H \to X$ is a unirational parameterization if $m_\alpha \geq N(\alpha, m_{\alpha-1})$ for $\alpha = 1, 2, ..., d-1$.

Let p be a general point on X, H_p be the fiber of $H \to X$ over p, Γ be the $(m+1)$-plane spanned by Λ and p, and

$$X \cap \Gamma = \Lambda + X_\Gamma, X_\Gamma \cap \Lambda = Y_\Gamma.$$

Obviously, on H_p, $\Gamma_{d-1} = \Gamma$ and Λ_{d-2} runs over all m_{d-2}-planes contained in Y_Γ. Hence there is a dominant map $H_p \to F_{m_{d-2}}(Y_\Gamma)$, whose generic fiber is the closure of the locus of pairs

$$(\Lambda_1, \Lambda_2, ..., \Lambda_{d-2}; \Gamma_1, \Gamma_2, ..., \Gamma_{d-2})$$

such that

$$\text{mult}_{\Lambda_i}(X \cap \Lambda_{i+1}) = d - 1 - i$$
$$\Lambda_i \text{ is a general } m_i\text{-plane in } \Lambda_{i+1}$$

and

$$\Gamma_i \text{ is a general } m_i\text{-plane containing } \Lambda_i \text{ and contained in } \Lambda_{i+1}$$

for $i = 1, ..., d-2$ with Λ_{d-2} the generic point of $F_{m_{d-2}}(Y_\Gamma)$. By [HMP, Theorem 4.1], the generic fiber of $H_p \to F_{m_{d-2}}(Y_\Gamma)$ is rational. And since $F_{m_{d-2}}(Y_\Gamma)$ is geometrically unirational when m_{d-1} is sufficiently large with respect to m_{d-2} and d, H_p is geometrically unirational when $m >> d$.

Using the circle method, Birch showed that a hypersurface over a number field satisfies the *nonsingular Hasse principle* [Lan, Chap. X] if the codimension of the singular locus of the hypersurface is sufficiently large with respect to the degree of the hypersurface [Bir]. Combining this with Corollary 2.1, we have

Theorem 2.3. *Let X be a degree d hypersurface over a number field. Then the* nonsingular Hasse principle *holds for the Fano variety $F_k(X)$ of X if the codimension of singular locus of X is sufficiently large with respect to both d and k.*

Hence we may combine Theorem 2.3 with [HMP, Theorem 4.1] to give a local criterion for a hypersurface over a number field to be unirational.

Theorem 2.4. *For any number d, there exist two numbers $n(d)$ and $k(d)$ such that any degree d hypersurface X over a number field K is unirational if* $\operatorname{codim} X_{sing} > n(d)$ *and there exists a number $k > k(d)$ such that the Fano variety $F_k(X)$ has a simple rational point over any completion of K.*

Theorem 2.3 answers a question raised by Mazur. He also suggested Theorem 2.4 assuming Theorem 2.3 holds.

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