

DENSITY OF RATIONAL CURVES ON $K3$ SURFACES

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ABSTRACT. Using the dynamics of self rational maps of elliptic $K3$ surfaces together with deformation theory, we prove that the union of rational curves is dense on a very general $K3$ surface and that the union of elliptic curves is dense in the 1st jet space of a very general $K3$ surface, both in the strong topology. The techniques developed here also lend themselves to applications to Abel-Jacobi images, and we explore some consequences in the Appendix.

1. INTRODUCTION

1.1. Density of rational curves. The main purpose of this note is to prove that the union of all rational curves on a “very general” projective $K3$ surface X is dense in the usual topology. Here “very general” takes some explanation. It is weaker than the usual sense of being in the complement of a countably many closed proper subvarieties.

Let \mathcal{K}_g be the moduli space of $K3$ surfaces of genus $g \geq 2$ and \mathcal{S}_g be the universal family over \mathcal{K}_g . That is,

$$(1.1) \quad \mathcal{K}_g = \left\{ (X, L) : X \text{ is a } K3 \text{ surface, } L \in \text{Pic}(X) \text{ is ample primitive} \right. \\ \left. \text{and } L^2 = 2g - 2 \right\}$$

and $\mathcal{S}_g = \{(X, L, p) : (X, L) \in \mathcal{K}_g, p \in X\}$.

Let $\mathcal{C}_{g,n} \subset \mathcal{S}_g$ be a closed subscheme of \mathcal{S}_g whose fiber over a general point $(X, L) \in \mathcal{K}_g$ is the union of all irreducible rational curves in the linear series $|nL| = \mathbb{P}H^0(X, nL)$. Our main theorem is

Theorem 1.1. *For all $g \geq 2$, the set*

$$(1.2) \quad \bigcup_{n=1}^{\infty} \mathcal{C}_{g,n}$$

is dense in \mathcal{S}_g .

Using an elementary topological argument, we can easily conclude the following (actually equivalent) statement.

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Corollary 1.2. *For all $g \geq 2$, the set*

$$(1.3) \quad \left\{ (X, L) \in \mathcal{K}_g : \bigcup_{n=1}^{\infty} \mathcal{C}_{X,nL} \text{ is not dense in } X \right\}$$

is of Baire the first category, i.e., a countable union of nowhere dense subsets in \mathcal{K}_g under the usual topology, where $\mathcal{C}_{X,nL}$ is the fiber of $\mathcal{C}_{g,n}$ over (X, L) . Hence the set of $K3$ surfaces of genus g whose rational curves are dense is of Baire the second category.

This partially answers a question raised in [C-L] (Conjecture 1.2), although we expect that the union of rational curves are dense on every projective $K3$ surface, not only the general ones. However, the method here does not lend itself to handle every projective $K3$ surface. On the other hand, it is unknown whether the union of rational curves is dense in the Zariski topology on every projective $K3$ surface.

Conjecture 1.3. *The union of rational curves is dense in the Zariski topology on every projective $K3$ surface X . That is, there are infinitely many rational curves on X .*

Remark 1.4. This was known for a very general $K3$ surface using a deformational argument [M-M]. However, to deal with every projective $K3$ surface, some new methods are needed. Recently, some substantial progress has been made on the conjecture. This was proved in [BHT] for $g = 2$ and $\text{Pic}(X) = \mathbb{Z}$ using characteristic p reduction. Their method was further developed in [L-L], where the conjecture was settled in all major cases with the only exception $\text{rank Pic}(X) = 2$. Also the conjecture was known for all elliptic $K3$ surfaces [B-T] (see also [H-T]). So the only unknown cases are $K3$ surfaces of Picard rank two which do not admit an elliptic fibration. However, their method does not seem to apply to the strong topology. Density of rational curves on $K3$ surfaces in both Zariski and strong topologies is related to Lang's conjecture on these surfaces [La].

Although we are unable to prove the density of rational curves on every $K3$ surface in the strong topology, we can do this for every elliptic $K3$ surface X of $\text{rank Pic}(X) \leq 19$ using the following result in the previously mentioned paper [B-T, Theorem 1.8]:

Theorem 1.5 (Bogomolov-Tschinkel). *Let X be an algebraic $K3$ surface with $\text{rank Pic}(X) \leq 19$ admitting an elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$. Then this fibration has infinitely many rational non torsion multisections. Here a rational non torsion multisection C of $\pi : X \rightarrow \mathbb{P}^1$ is an irreducible rational curve $C \subset X$ such that C meets the general fiber X_b of π at (at least) one point p satisfying $L - mp \notin J(X_b)_{\text{tors}}$, where L is an ample line bundle on X , $m = L \cdot X_b$ and $J(X_b) = \text{Pic}^0(X_b)$ is the Jacobian of the elliptic curve X_b .*

It follows that rational curves are Zariski dense on every elliptic $K3$ surface [B-T]. To conclude that rational curves are dense in the strong topology on every elliptic surface X with $\text{rank Pic}(X) \leq 19$, we need a classic lemma of H. Weyl (Lemma 2.5) and a study of normal functions associated to an elliptic fibration. We are able to prove that

Theorem 1.6. *The union of rational curves is dense in the strong topology on every elliptic $K3$ surface X with $\text{rank Pic}(X) \leq 19$.*

We want to point out that density of rational curves on elliptic $K3$ surfaces does not imply the same on a general $K3$ surface directly despite the fact that elliptic $K3$ surfaces are dense in the moduli space of $K3$ surfaces since a rational non torsion multisection does not deform to a rational curve on a general $K3$ unless it is a multiple of the polarization divisor.

1.2. Density of elliptic curves. For convenience, we will call a point *Baire general* if it lies in the complement of a countable union of nowhere dense subsets.

Of course, every $K3$ surface X is covered by one-parameter families of elliptic curves. It is natural to ask whether these curves are dense when lifted to the first jet space $\mathbb{P}T_X$ of X . Here the lifting $df : C \dashrightarrow \mathbb{P}T_X$ of a map $f : C \rightarrow X$ is induced by the map $f_* : T_C \rightarrow f^*T_X$ on the tangent sheaves.

For every $n \in \mathbb{Z}^+$, we let $\mathcal{W}_{g,n}$ be the closure of the subscheme of $\mathbb{P}H^0(\mathcal{S}_g, nL)$ whose fiber over a general (X, L) consists of irreducible elliptic curves in $|nL|$ and let

$$(1.4) \quad \mathcal{E}_{g,n} = \{(X, L, E, p) : (X, L, E) \in \mathcal{W}_{g,n}, p \in E\} \subset \mathcal{W}_{g,n} \times_{\mathcal{K}_g} \mathcal{S}_g$$

be the universal family over $\mathcal{W}_{g,n}$.

Theorem 1.7. *Let $\varphi : \mathcal{E}_{g,n} \dashrightarrow \mathbb{P}T_{\mathcal{S}_g/\mathcal{K}_g}$ be the rational map induced by the map*

$$(1.5) \quad T_{\mathcal{E}_{g,n}/\mathcal{W}_{g,n}} \rightarrow T_{\mathcal{S}_g/\mathcal{K}_g}$$

on the relative tangent sheaves. Then

$$(1.6) \quad \bigcup_{n=1}^{\infty} \varphi(\mathcal{E}_{g,n})$$

is dense in $\mathbb{P}T_{\mathcal{S}_g/\mathcal{K}_g}$ for all $g \geq 2$, where $\varphi(\mathcal{E}_{g,n})$ is the proper transform of $\mathcal{E}_{g,n}$ under φ .

It follows that the union of $\varphi(\mathcal{E}_{X,nL})$ is dense in $\mathbb{P}T_X$ for a Baire general $(X, L) \in \mathcal{K}_g$, where $\mathcal{E}_{X,nL}$ is the fiber of $\mathcal{E}_{g,n}$ over the point $(X, L) \in \mathcal{K}_g$.

1.3. Hyperbolic geometry of $K3$ surfaces. One of the reasons we are interested in the elliptic curves on a $K3$ surface X comes from the fact that they are the images of holomorphic maps $\mathbb{C} \rightarrow X$. So they are closely related to the hyperbolic geometry of X . Recall the definition of *Kobayashi-Royden* (KR) pseudo-metric on a complex manifold X (cf. [K]): for a point $p \in X$ and a nonzero tangent vector $v \in T_{X,p}$, we define

$$(1.7) \quad \|v\|_\kappa = \inf\{\lambda > 0 : \exists \text{ a holomorphic map } f : \Delta \rightarrow X \\ \text{with } f(0) = p, f_*(\partial/\partial z) = \lambda^{-1}v\}$$

Obviously, if there is a holomorphic $f : \mathbb{C} \rightarrow X$ such that $f(0) = p$ and $f_*(\partial/\partial z) = v$ for some tangent vector $v \in T_{X,p}$, then $\|v\|_\kappa = 0$. In particular, if there is holomorphic dominant map $f : \mathbb{C}^n \rightarrow X$, then the KR pseudo-metric vanishes everywhere on X . In [B-L], G. Buzzard and S. Y. Lu classified all the algebraic surfaces that are holomorphically dominable by \mathbb{C}^2 . They settled every single case except $K3$ surfaces, for which they proved all elliptic and Kummer $K3$ surfaces can be holomorphically dominated by \mathbb{C}^2 . But it is unknown whether a general $K3$ surface can be dominated by \mathbb{C}^2 or has everywhere vanishing KR pseudo-metric, although this is expected to be true.

Conjecture 1.8 (Buzzard-Lu). *Every complex $K3$ surface is holomorphically dominable by \mathbb{C}^2 . As a consequence, it has everywhere vanishing KR pseudo-metric.*

By Theorem 1.7, we know at least that the following holds.

Corollary 1.9. *For $g \geq 2$, a Baire general $(X, L) \in \mathcal{K}_g$ and a Baire general $p \in X$, the set $\{v \in T_{X,p} : \|v\|_\kappa = 0\}$ is dense in $T_{X,p}$.*

The layout of this paper is as follows. We will prove our main theorems in Sec. 2. In Sec. 3, we will re-interpret a key step of our proof in terms of Poincaré normal functions and prove Theorem 1.6. Finally in the Appendix, we consider applications of our techniques to images of Abel-Jacobi maps. The main results of the Appendix are stated in Theorem 4.1 and Corollaries 4.2 and 4.4.

2. PROOFS OF THEOREM 1.1 AND 1.7

2.1. Elliptic $K3$ surfaces. Our strategy is to show that rational curves are dense on X for (X, L) in a dense subset of \mathcal{K}_g . Then Theorem 1.1 will follow easily. It is well known that Kummer surfaces are dense in the moduli space of polarized $K3$ surfaces (cf. [BPV]). This implies that polarized elliptic $K3$ surfaces are dense. An elliptic $K3$ surface X is a $K3$ surface with an elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$. A general projective elliptic $K3$ surface has Picard lattice given by

$$(2.1) \quad \begin{bmatrix} 2g-2 & m \\ m & 0 \end{bmatrix}$$

where m is a positive integer. That is, the Picard group $\text{Pic}(X)$ of X is generated by effective classes L and F satisfying

$$(2.2) \quad L^2 = 2g - 2, LF = m \text{ and } F^2 = 0$$

and the elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$ is given by the pencil $|F|$.

Let $\mathcal{P}_{g,m} \subset \mathcal{K}_g$ be the subscheme consisting of $K3$ surfaces whose Picard lattices contain (2.1) as primitive sublattices and that are polarized by L . The general theory of $K3$ surfaces tells us that $\mathcal{P}_{g,m}$ is irreducible of codimension 1 in \mathcal{K}_g for each pair (g, m) . Also the union of $\mathcal{P}_{g,m}$ is dense in \mathcal{K}_g , as mentioned above. For our purpose, we need the following slightly stronger statement.

Lemma 2.1. *The union*

$$(2.3) \quad \bigcup_{2|m} \mathcal{P}_{g,m}$$

is dense in \mathcal{K}_g for all $g \geq 2$.

Proof. It suffices to show that the union of $\mathcal{P}_{g,m}$ contains all polarized Kummer surfaces for m even. Let X be a Kummer surface with $\text{Pic}(X)$ generated by effective classes N , E_i and $M = \frac{1}{2} \sum E_i$ satisfying

$$(2.4) \quad N^2 = 2n > 0, E_i^2 = -2 \text{ and } NE_i = E_i E_j = 0$$

for $0 \leq i \neq j \leq 15$.

Suppose that X is principally polarized by

$$(2.5) \quad L = aN - \sum_{i=0}^{15} a_i E_i = aN - 2a_0 M - \sum_{i=1}^{15} (a_i - a_0) E_i$$

where $a \in \mathbb{Z}^+$ and $a_i > 0$ are all either integers or half-integers satisfying

$$(2.6) \quad \gcd(a, 2a_0, a_i - a_0) = 1 \text{ and } a^2 n = g - 1 + \sum_{i=0}^{15} a_i^2.$$

Since every positive integer can be written as the square sum of four integers, we may let

$$(2.7) \quad F = bN - \sum_{i=1}^5 b_i E_i$$

where $b \in \mathbb{Z}^+$ and $b_i \in \mathbb{Z}_{\geq 0}$ satisfy

$$(2.8) \quad b^2 n = \sum_{i=1}^5 b_i^2.$$

It remains to choose b and b_i such that $m = LF$ is even and the sublattice generated by L and F is primitive.

We let $b_1 = 1$. Then the sublattice generated by L and F is primitive if and only if

$$(2.9) \quad \gcd(a - (a_1 - a_0)b, 2a_0, (a_i - a_0) - (a_1 - a_0)b_i)_{i \geq 2} = 1$$

which holds if we choose b such that

$$(2.10) \quad \gcd(a - (a_1 - a_0)b, 2a_0) = \gcd(a, a_1 - a_0, 2a_0).$$

This is easy to do but we need to guarantee that m is even at the same time.

When all a_i are integers, we simply choose b such that (2.10) holds. When all a_i are half integers, we may choose $2|b$ such that (2.10) holds. In both cases, it is easy to check that $2|m$. \square

Remark 2.2. The choice of m being even is purely technical. As we will see, it simplifies the construction of the degeneration of elliptic $K3$ surfaces. It could be removed at the cost of making our later argument more complicated.

2.2. Dynamics under self rational maps. An elliptic $K3$ surface admits self rational maps induced by fiberwise elliptic curve endomorphism (cf. [D]).

Let $(X, L) \in \mathcal{P}_{g,m}$. Fixing $A \in \text{Pic}(X)$ with $AF = a$, we can construct a rational map $\phi_A : X \dashrightarrow X$ by sending a point p lying on a smooth fiber $X_q = \pi^{-1}(q)$ to the point $A - (a - 1)p$ on X_q using the group structure of the elliptic curve X_q , by which we mean that we send p to the unique point $p' \in X_q$ given by

$$(2.11) \quad A|_{X_q} \sim_{\text{rat}} (a - 1)p + p'$$

on X_q . Obviously, ϕ_A is dominant unless $a = 1$. Of course, this construction works for all fibrations of abelian varieties, not just elliptic $K3$'s.

Let $C \subset X$ be an irreducible rational curve which is not contained in a fiber of π . The proper transform $\phi_A(C)$ of C under ϕ_A is also an irreducible rational curve on X not contained in a fiber. Naturally, we expect the following to be true.

Proposition 2.3. *For all $g, m \in \mathbb{Z}^+$ satisfying $g \geq 2$ and $2|m$ and a Baire general $(X, L) \in \mathcal{P}_{g,m}$, there exists an irreducible rational curve $C \subset X$ such that the set*

$$(2.12) \quad \bigcup_{A \in \text{Pic}(X)} \phi_A(C) = \bigcup_{n \in \mathbb{Z}} \phi_{nL}(C)$$

is dense on X .

We cannot yet conclude Theorem 1.1 from Proposition 2.3 since $\phi_A(C)$ may not lie on the fiber $\mathcal{C}_{g,n}$ over the point (X, L) . Indeed, if $C \in |aL + bF|$, $\phi_{kL}(C) \in |aL + b_kF|$ for some $b_k \in \mathbb{Z}$. As $|k| \rightarrow \infty$, $b_k \rightarrow \infty$ since we have only finitely many rational curves in each linear series. Hence $\phi_{kL}(C) \not\sim_{\text{rat}} nL$ for all $n \in \mathbb{Z}$, when $|k|$ is sufficiently large. So the rational curve $\phi_{kL}(C)$ alone cannot be deformed to a rational curve on a general $K3$ surface. But we can find a rational curve $B_k \subset X$ such that $B_k + \phi_{kL}(C) \sim_{\text{rat}} nL$ for

some $n \in \mathbb{Z}$ and the union $B_k \cup \phi_{kL}(C)$ can be deformed to an irreducible rational curve on a general K3 surface. Namely, we can prove the following.

Proposition 2.4. *For all $g, m \in \mathbb{Z}^+$ satisfying $g \geq 2$ and $2|m$, a general $(X, L) \in \mathcal{P}_{g,m}$ and an irreducible rational curve $C \subset X$ such that $C \not\sim_{\text{rat}} lL$ for all $l \in \mathbb{Z}$,*

- *there exists an irreducible rational curve $B \subset X$ such that $B \cup C$ lies on an irreducible component of $\mathcal{C}_{g,n}$ that dominates \mathcal{K}_g ;*
- *there exists an irreducible elliptic curve $B \subset X$ such that $B \cup C$ lies on an irreducible component of $\mathcal{E}_{g,n}$ that dominates \mathcal{S}_g .*

Clearly, Proposition 2.3 and 2.4 together will give us Theorem 1.1 and 1.7.

Let X_q be a general fiber π and $p \in X_q \cap C$. Then ϕ_{nL} sends p to the point

$$(2.13) \quad \phi_{nL}(p) = nL - (mn - 1)p$$

and hence

$$(2.14) \quad \phi_{nL}(p) - p = n(\phi_L(p) - p) = n(L - mp)$$

in the Jacobian $\text{Pic}^0(X_q) = J(X_q)$ of the elliptic curve X_q .

Proposition 2.3 will follow if we can prove that the subgroup of $J(X_q)$ generated by $L - mp$ is dense. So we naturally ask which points on an elliptic curve, or more generally a compact complex torus, generate a dense subgroup. This is an elementary yet interesting problem in itself, answered by the following lemma which can be traced back to H. Weyl. But since we cannot locate a definite reference, we will supply a proof here.

Lemma 2.5. *Let $A = \mathbb{R}^n / \mathbb{Z}^n$ be a compact real torus of dimension n . For a point $p = (x_1, x_2, \dots, x_n) \in A$, $\mathbb{Z}p = \{kp : k \in \mathbb{Z}\}$ is dense in A if and only if $1, x_1, x_2, \dots, x_n$ are linearly independent on \mathbb{Q} . In particular, the set*

$$(2.15) \quad \{p \in A : \mathbb{Z}p \text{ is not dense in } A\}$$

is of Baire the first category.

Proof. If $1, x_1, x_2, \dots, x_n$ are linearly dependent on \mathbb{Q} , then the group generated by p and \mathbb{Z}^n in \mathbb{R}^n lies in the union of hyperplanes

$$(2.16) \quad \bigcup_{a \in \mathbb{Z}} \{a_1 x_1 + a_2 x_2 + \dots + a_n x_n + a = 0\}$$

for some integers a_i not all zero. Clearly, the set (2.16) is not dense.

On the other hand, suppose that $1, x_1, x_2, \dots, x_n$ are linearly independent on \mathbb{Q} . We prove by induction on n . This is obvious when $n = 1$.

Note that the closure $\overline{\mathbb{Z}p}$ of $\mathbb{Z}p$ is also a subgroup of A . By induction hypothesis, the group $\mathbb{Z}(\pi_k(p))$ is dense in $\mathbb{R}^{n-1} / \mathbb{Z}^{n-1}$ under the k -th projection $\pi_k : \mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{R}^{n-1} / \mathbb{Z}^{n-1}$ given by

$$(2.17) \quad \pi_k(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, \hat{x}_k, \dots, x_n)$$

for $1 \leq k \leq n$. So it suffices to show that $\overline{\mathbb{Z}p}$ contains the line

$$(2.18) \quad L_k = \{(0, 0, \dots, x_k, 0, \dots, 0)\} \subset A$$

for some k .

Since $\mathbb{Z}(\pi_1(p))$ is dense in $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$, there exists a point

$$(2.19) \quad p_m = \left(r_m, \frac{1}{m}, 0, 0, \dots, 0 \right) \in \overline{\mathbb{Z}p}$$

for every positive integer m . If $r_m \notin \mathbb{Q}$ for some m , then $\mathbb{Z}(mp_m)$ is dense in L_1 and we are done. Therefore, $r_m \in \mathbb{Q}$ for all m and we write $r_m = a_m/b_m$ for $a_m, b_m \in \mathbb{Z}$ and $\gcd(a_m, b_m) = 1$.

Let $c_m = \gcd(m, b_m)$. Then by Chinese remainder theorem, $\mathbb{Z}p_m$ consists of all points

$$(2.20) \quad \mathbb{Z}p_m = \left\{ \left(\frac{\alpha}{b_m}, \frac{\beta}{m}, 0, \dots, 0 \right) : \alpha, \beta \in \mathbb{Z}, \alpha \equiv a_m \beta \pmod{c_m} \right\}.$$

Therefore, if either $\limsup c_m/b_m = 0$ or $\limsup c_m/m = 0$, $\overline{\mathbb{Z}p}$ contains either L_1 or L_2 and we are done. Therefore, $(c_m/b_m, c_m/m)$ only takes finitely many different values. Consequently, there exist coprime integers u and v such that $b_m = uc_m$ and $m = vc_m$ for infinitely many $m \in \mathbb{Z}^+$. This also implies that $|c_m| \rightarrow \infty$.

Since $\gcd(u, v) = \gcd(u, a_m) = 1$, $\gcd(u, a_m v) = 1$ and hence there exist integers s and t such that $a_m vs + ut = 1$. Let us consider the linear map $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by

$$(2.21) \quad \phi(x_1, x_2, x_3, \dots, x_n) = (sx_1 + tx_2, ux_1 - a_m vx_2, x_3, \dots, x_n).$$

Clearly, ϕ is an isomorphism on \mathbb{Z}^n and hence induces an automorphism of A . We see that $\phi(p_m) = (1/(uvc_m), 0, 0, \dots, 0)$ for infinitely many $m \in \mathbb{Z}^+$. And since $|c_m| \rightarrow \infty$, $\overline{\mathbb{Z}\phi(p)}$ contains L_1 and we are done. \square

There are two ways we can show that $L - mp$ generates a dense subgroup of $J(X_q)$ using the above lemma. One way is via normal functions. This will be done in Sec. 3. The other way is to show that $L - mp$ is general in $J(X_q)$ as X and q vary.

First of all, we have to make what we mean by ‘‘general in $J(X_q)$ ’’ precise. Let

$$(2.22) \quad \mathcal{S}_{g,m} = \mathcal{S}_g \times_{\mathcal{K}_g} \mathcal{P}_{g,m}$$

be the pullback of the universal family \mathcal{S}_g to $\mathcal{P}_{g,m} \subset \mathcal{K}_g$ and let $\mathcal{C}_{g,m,A} \subset \mathcal{S}_{g,m}$ be the closed subscheme whose fiber over a general point $(X, L) \in \mathcal{P}_{g,m}$ is the union of all irreducible rational curves in $|A|$, where $A \in \text{Pic}(\mathcal{S}_{g,m}/\mathcal{P}_{g,m})$. Note that we have an elliptic fibration

$$(2.23) \quad \pi : \mathcal{S}_{g,m} \rightarrow \mathbb{P}^1 \times \mathcal{P}_{g,m}$$

given by the pencil $|F|$. The induced map $\mathcal{C}_{g,m,A} \rightarrow \mathbb{P}^1 \times \mathcal{P}_{g,m}$ is generically finite if it is dominant.

Let $p \in \mathcal{C}_{g,m,A}$ be a point over a general point $q = \pi(p) \in \mathbb{P}^1 \times \mathcal{P}_{g,m}$. We have a map $p \rightarrow J(E)$ by sending p to $L - mp$, where $E = \pi^{-1}(q) \subset \mathcal{S}_{g,m}$ is the fiber of π over q . Note that $J(E) = \text{Pic}^0(E)$ is the elliptic curve E with a base point 0 corresponding to the trivial bundle \mathcal{O}_E . So we have two marked points $(0, L - mp)$ on $J(E)$. Namely, we have a well-defined map

$$(2.24) \quad \gamma : \mathcal{C}_{g,m,A} \rightarrow \overline{\mathcal{M}}_{1,2}$$

sending p to $(J(E), 0, L - mp)$, where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves of genus g with n marked points. By saying $L - mp$ is general, we simply mean that γ is dominant.

Lemma 2.6. *For all $g, m \in \mathbb{Z}^+$ satisfying $g \geq 2$ and $2|m$, there exists an irreducible component of $\mathcal{C}_{g,m,A}$ dominating $\mathbb{P}^1 \times \mathcal{P}_{g,m}$ via π and dominating $\overline{\mathcal{M}}_{1,2}$ via γ for some $A \in \text{Pic}(\mathcal{S}_{g,m}/\mathcal{P}_{g,m})$.*

This, together with Lemma 2.5, will give us Proposition 2.3.

If $\mathcal{C}_{g,m,A}$ dominates $\mathbb{P}^1 \times \mathcal{P}_{g,m}$, it obviously dominates $\overline{\mathcal{M}}_{1,1}$ by

$$(2.25) \quad \mathcal{C}_{g,m,A} \xrightarrow{\gamma} \overline{\mathcal{M}}_{1,2} \xrightarrow{\tau} \overline{\mathcal{M}}_{1,1}$$

where τ is the forgetting map. So to show that γ is dominant, it suffices to show that the closure of the image of γ contains the boundary component $\overline{\mathcal{M}}_{0,4} \subset \overline{\mathcal{M}}_{1,2}$. The proof of this fact relies on a degeneration argument.

2.3. Deformation of $K3$ surfaces. Following the idea in [CLM], we can deform a $K3$ surface to a union of two rational surfaces. Let $R = R_1 \cup R_2$ be the union of two smooth rational surfaces R_1 and R_2 meeting transversely along a smooth elliptic curve $D = R_1 \cap R_2$ where $D = -K_{R_i}$ in $\text{Pic}(R_i)$ for $i = 1, 2$. We see that R is simply connected and the dualizing sheaf ω_R of R is trivial. So it is expected that R can be deformed to a $K3$ surface. The deformation of R is governed by the map

$$(2.26) \quad \begin{aligned} \text{Ext}(\Omega_R, \mathcal{O}_R) &\rightarrow H^0(T^1(R)) = H^0(\text{Ext}(\Omega_R, \mathcal{O}_R)) \\ &= H^0(\mathcal{O}_D(-K_{R_1} - K_{R_2})). \end{aligned}$$

Then R can be deformed to a $K3$ surface only if the image of the above map is base point free in $H^0(T^1(R))$. That is, $R_{\text{sing}} = D$ can be smoothed when R deforms. This puts some restrictions on R_i . A necessary condition is that $\mathcal{O}_D(-K_{R_1} - K_{R_2})$ is base point free. It can be guaranteed if we choose R_i to be Fano.

A deformation of R is a complex $K3$ surface, not necessarily projective. In order to deform R to a projective $K3$ surface, in particular, to deform R to a $K3$ surface in $\mathcal{P}_{g,m}$, we need to construct R in such a way that it has two line bundles L and F satisfying (2.2). Let $L_i = L|_{R_i}$ and $F_i = F|_{R_i}$ for $i = 1, 2$. Then

$$(2.27) \quad L_1 \Big|_D = L_2 \Big|_D \quad \text{and} \quad F_1 \Big|_D = F_2 \Big|_D.$$

Indeed, R is constructed by gluing R_1 and R_2 transversely along D such that

$$(2.28) \quad e_1^*L_1 = e_2^*L_2 \text{ and } e_1^*F_1 = e_2^*F_2$$

where e_i is the embedding $D \hookrightarrow R_i$ for $i = 1, 2$.

As in [CLM] and [C], we can degenerate every $K3$ surface of genus $g \geq 3$ to a union $R = R_1 \cup R_2$ as follows:

- if $g \geq 3$ is odd, we let $R_i \cong \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $R_1 \cup R_2$ be polarized by the ample line bundle L where

$$(2.29) \quad L_i = L \Big|_{R_i} = M_i + \frac{g-1}{2}G_i$$

with M_i and G_i being the generators of $\text{Pic}(R_i)$ satisfying

$$(2.30) \quad M_i^2 = G_i^2 = 0 \text{ and } M_iG_i = 1$$

for $i = 1, 2$;

- if $g \geq 4$ is even, we let $R_i \cong \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and $R_1 \cup R_2$ be polarized by the ample line bundle L where

$$(2.31) \quad L_i = L \Big|_{R_i} = M_i + \frac{g}{2}G_i$$

with M_i and G_i being the generators of $\text{Pic}(R_i)$ satisfying

$$(2.32) \quad -M_i^2 = M_iG_i = 1 \text{ and } G_i^2 = 0$$

for $i = 1, 2$.

Note that this does not cover the case $g = 2$. The genus 2 case will be treated separately in 2.5.

Such R can be deformed to a general $K3$ surface in \mathcal{K}_g . In order to deform it to a $K3$ surface in $\mathcal{P}_{g,m}$, we need to have another line bundle $F \in \text{Pic}(R)$ besides $L \in \text{Pic}(R)$. Here we simply let

$$(2.33) \quad F_i = F \Big|_{R_i} = \frac{m}{2}G_i$$

with the “tricky” requirement that

$$(2.34) \quad \mathcal{O}_D(G_1 - G_2) \in \text{Pic}^0(D) = J(D) \text{ is an } (m/2)\text{-torsion.}$$

We glue R_1 and R_2 in such a way that (2.27) are the only relations between $\text{Pic}(R_i)$, with L_i and F_i given by (2.29), (2.31) and (2.33), respectively. More precisely, the kernel $\text{Pic}(R)$ of the map

$$(2.35) \quad \text{Pic}(R_1) \oplus \text{Pic}(R_2) \xrightarrow{e_1^* - e_2^*} \text{Pic}(D)$$

is freely generated by $L = L_1 \oplus L_2$ and $F = F_1 \oplus F_2$. Numerically, we have

$$(2.36) \quad L_i^2 = g - 1, L_iF_i = \frac{m}{2} \text{ and } F_i^2 = 0.$$

Remark 2.7. It may appear that F is not primitive by (2.33). It actually is since $G_1 - G_2$ is a torsion point of $J(D)$ of order $m/2$ and hence there does not exist $k \in \mathbb{Z}$ such $\mathcal{O}_D(kG_1) = \mathcal{O}_D(kG_2)$ unless $(m/2)|k$. It may also appear that $h^0(F) = m/2 + 1$ by (2.33). Actually, $h^0(F) = 2$, i.e., $|F|$ is a pencil, again by (2.34). Indeed, a member of $|F|$ is a union $N_1 \cup N_2 \cup \dots \cup N_m$ where

- $N_k \subset R_1$ and $N_k \in |G_1|$ for k odd and $N_k \subset R_2$ and $N_k \in |G_2|$ for k even;
- $\cup N_k$ meets D at points q_1, q_2, \dots, q_m such that

$$(2.37) \quad N_k \cdot D = q_k + q_{k+1}$$

for $1 \leq k \leq m$, where we let $q_{m+1} = q_1$.

Obviously, such a union $\cup N_k$ moves in a base point free pencil.

Such R can be deformed to $K3$ surfaces in $\mathcal{P}_{g,m}$. That is, there exists a one-parameter families \mathcal{S} over the disk $\Delta = \{|t| < 1\}$ and two line bundles L and $F \in \text{Pic}(\mathcal{S}/\Delta)$ such that $(\mathcal{S}_t, L) \in \mathcal{P}_{g,m}$ for $t \neq 0$ and $\mathcal{S}_0 = R$ is the union R with L and F constructed as above. The proofs of Lemma 2.6 and Proposition 2.4 both depend on the construction of certain rational curves on the general fibers \mathcal{S}_t . Our strategy is to produce a reducible rational curve on the central fiber R , called a limiting rational curve in [C], and show that it can be deformed to an irreducible rational curve on the general fibers.

Lemma 2.8. *For all $g, m \in \mathbb{Z}^+$ satisfying $g \geq 2$ and $2|m$ and a general $(X, L) \in \mathcal{P}_{g,m}$, there is an irreducible nodal rational curve in $|aL + bF|$ for all $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}$ satisfying*

$$(2.38) \quad \max \left(2a \left\lfloor \frac{g-1}{2} \right\rfloor, a \right) + bm > 0$$

and $b^2 + (g-2)^2 \neq 0$.

Proof (when $g \geq 3$). Our construction of limiting rational curves $C_1 \cup C_2$ with $C_i \subset R_i$ is very similar to the construction in [C], but with some added difficulties. Namely, we have to make sure that

$$(2.39) \quad \begin{aligned} &\text{there does not exist } C'_1 \cup C'_2 \subsetneq C_1 \cup C_2 \\ &\text{such that } C'_1 \cup C'_2 \in |a'C + b'F| \text{ for some } a', b' \in \mathbb{Z}; \end{aligned}$$

otherwise, a deformation of $C_1 \cup C_2$ onto a general fiber \mathcal{S}_t is not necessarily irreducible. This is a little trickier here due to the fact $\text{rank Pic}(R) = 2$ and the condition (2.34).

The one-parameter family \mathcal{S} has sixteen rational double points p_1, p_2, \dots, p_{16} lying on D , which are precisely the zeros of a section in $H^0(T^1(R))$ that is in turn the image of the Kodaira-Spencer class of \mathcal{S}/Δ under the map (2.26).

So these sixteen points satisfy

$$(2.40) \quad \begin{aligned} \mathcal{O}_D(p_1 + p_2 + \dots + p_{16}) &= \mathcal{O}_D(-K_{R_1} - K_{R_2}) \\ &= \begin{cases} \mathcal{O}_D(2M_1 + 2G_1 + 2M_2 + 2G_2) & \text{if } 2 \nmid g \\ \mathcal{O}_D(2M_1 + 3G_1 + 2M_2 + 3G_2) & \text{if } 2 \mid g \end{cases} \end{aligned}$$

and this is the only relation among p_1, p_2, \dots, p_{16} for a general choice of \mathcal{S} .

We write

$$(2.41) \quad (aL + bF) \Big|_{R_i} = aM_i + \left(a \left\lfloor \frac{g}{2} \right\rfloor + \frac{bm}{2} \right) G_i = aM_i + lG_i.$$

Case $2 \nmid g$ and $a \leq l$. We let

$$(2.42) \quad C_i = I_{i1} \cup I_{i2} \cup \dots \cup I_{i,a-1} \cup J_{i1} \cup J_{i2} \cup \dots \cup J_{i,a-1} \cup \Gamma_i$$

be the curve on R_i ($i = 1, 2$) with irreducible components $I_{ij} \in |G_i|$, $J_{ij} \in |M_i|$ and $\Gamma_i \in |M_i + (l - a + 1)G_i|$ given by

$$(2.43) \quad \begin{aligned} I_{11} \cdot D &= p_1 + q_1, & J_{21} \cdot D &= q_1 + q_2 \\ I_{12} \cdot D &= q_2 + q_3, & J_{22} \cdot D &= q_3 + q_4 \\ &\dots & & \\ I_{1,a-1} \cdot D &= q_{2a-4} + q_{2a-3}, & J_{2,a-1} \cdot D &= q_{2a-3} + q_{2a-2} \end{aligned}$$

$$(2.44) \quad \begin{aligned} I_{21} \cdot D &= p_2 + r_1, & J_{11} \cdot D &= r_1 + r_2 \\ I_{22} \cdot D &= r_2 + r_3, & J_{12} \cdot D &= r_3 + r_4 \\ &\dots & & \\ I_{2,a-1} \cdot D &= r_{2a-4} + r_{2a-3}, & J_{1,a-1} \cdot D &= r_{2a-3} + r_{2a-2} \end{aligned}$$

and

$$(2.45) \quad \Gamma_1 \cdot D = p_2 + q_{2a-2} + (2l - 2a + 2)s, \quad \Gamma_2 \cdot D = p_1 + r_{2a-2} + (2l - 2a + 2)s$$

where q_j, r_j and s are points on D and we let $q_0 = p_1$ and $r_0 = p_2$. Intuitively, $C_1 \cup C_2$ is the union of two chains of curves, one starting at p_1 and the other starting at p_2 , consisting of curves in $|G_i|$ and $|M_i|$ alternatively and finally “joined” by Γ_1 and Γ_2 . We see that (2.39) holds because p_1 and p_2 are two general points on D and G_i and M_{3-i} are linearly independent in $\text{Pic}_{\mathbb{Q}}(D)$ for each $i = 1, 2$.

Case $2 \nmid g$ and $a > l$. Note that (2.38) implies $l > 0$ when g is odd. We use the same construction as above for $a \leq l$ by simply switching G_i and M_i and switching a and l .

Case $2 \mid g$. Note that $l > a$ by (2.38) when g is even. Let

$$(2.46) \quad \alpha = \left\lfloor \frac{a}{2} \right\rfloor \quad \text{and} \quad \beta = \left\lfloor \frac{a-1}{2} \right\rfloor$$

and let

$$(2.47) \quad C_i = I_{i1} \cup I_{i2} \cup \dots \cup I_{i\alpha} \cup J_{i1} \cup J_{i2} \cup \dots \cup J_{i\beta} \cup \Gamma_i$$

be the curve on R_i ($i = 1, 2$) with irreducible components $I_{ij}, J_{ik} \in |M_i + G_i|$ and $\Gamma_i \in |M_i + (l - a + 1)G_i|$ given by

$$\begin{aligned}
(2.48) \quad & I_{11} \cdot D = p_1 + p_3 + q_1, \quad I_{21} \cdot D = p_2 + q_1 + q_2 \\
& I_{12} \cdot D = p_1 + q_2 + q_3, \quad I_{22} \cdot D = p_2 + q_3 + q_4 \\
& \dots \\
(2.49) \quad & I_{1\alpha} \cdot D = p_1 + q_{2\alpha-2} + q_{2\alpha-1}, \quad I_{2\alpha} \cdot D = p_2 + q_{2\alpha-1} + q_{2\alpha}, \\
& J_{21} \cdot D = p_1 + p_4 + r_1, \quad J_{11} \cdot D = p_2 + r_1 + r_2 \\
& J_{22} \cdot D = p_1 + r_2 + r_3, \quad J_{12} \cdot D = p_2 + r_3 + r_4 \\
& \dots \\
& J_{2\beta} \cdot D = p_1 + r_{2\beta-2} + r_{2\beta-1}, \quad J_{1\beta} \cdot D = p_2 + r_{2\beta-1} + r_{2\beta},
\end{aligned}$$

and

$$\begin{aligned}
(2.50) \quad & \Gamma_1 \cdot D = p_4 + q_{2\alpha} + (\alpha - \beta)p_2 + (2l - 2a - \alpha + \beta + 1)s, \\
& \Gamma_2 \cdot D = p_3 + r_{2\beta} + (\alpha - \beta)p_1 + (2l - 2a - \alpha + \beta + 1)s
\end{aligned}$$

where q_j, r_k and s are points on D and we let $q_0 = p_3$ and $r_0 = p_4$. Since p_1, p_2, p_3, p_4 are in general position on D , it is not hard to see that (2.39) holds.

The curve $C_1 \cup C_2$ constructed above has the following properties in addition to (2.39):

- every component of C_i is a smooth rational curve and C_i has simple normal crossing outside of D ;
- if C_i and D meet at a point $q \notin \{p_1, p_2, \dots, p_{16}\}$, there is only one branch of C_i locally at q , i.e., C_i is smooth at q ;
- if C_i and D meet at a point $q \in \{p_1, p_2, \dots, p_{16}\}$, all local branches of C_i at q meet transversely with each other and also transversely with D .

Then by the argument in [C], $C_1 \cup C_2$ can be deformed to an irreducible nodal rational curve on the general fibers of \mathcal{S}/Δ . More precisely, there exists a flat family of curves $\mathcal{C} \subset \mathcal{S}$, after a base change, such that $\mathcal{C}_0 = C_1 \cup C_2$ and \mathcal{C}_t is an irreducible rational curve with only ordinary double points as singularities for $t \neq 0$. \square

Remark 2.9. The condition (2.38) is trivially satisfied when we take $a \gg |b|$. Therefore, for every $C \not\sim_{\text{rat}} lL$, there is an irreducible nodal rational curve in $|nL - C|$ for n sufficiently large. This is what we need for Proposition 2.4.

Now we are ready to prove Lemma 2.6.

Proof of Lemma 2.6 (when $g \geq 3$). By Lemma 2.8, there is an irreducible component of $\mathcal{C}_{g,m,A}$ dominating $\mathbb{P}^1 \times \mathcal{P}_{g,m}$, by setting e.g. $A = 2L + F$.

Let \mathcal{S}/Δ be the family of $K3$ surfaces constructed above. One may think of \mathcal{S} as the pullback of $\mathcal{S}_{g,m}$ under a map $\Delta^* \rightarrow \mathcal{P}_{g,m}$. Let $\mathcal{C} \subset \mathcal{S}$ be a family of rational curves constructed in the proof of Lemma 2.8 with $\mathcal{C}_t \in |2L + F|$.

One may think of \mathcal{C} as an irreducible component of the pullback of $\mathcal{C}_{g,m,2L+F}$ to \mathcal{S} . Correspondingly, we pull back the map γ to \mathcal{C} , i.e.,

$$(2.51) \quad \gamma : \mathcal{C} \rightarrow \overline{\mathcal{M}}_{1,2}$$

sending $p \in \mathcal{C}_t$ to $(J(E_p), 0, L - mp)$, where E_p is the fiber of the projection $\pi : \mathcal{S} \rightarrow \mathbb{P}^1 \times \Delta$ over the point $\pi(p)$.

It is enough to prove that

$$(2.52) \quad \dim(\gamma(\mathcal{C}) \cap \overline{\mathcal{M}}_{0,4}) = 1$$

where we think of $\overline{\mathcal{M}}_{0,4}$ as a component of $\overline{\mathcal{M}}_{1,2} \setminus \mathcal{M}_{1,2}$. Instead of directly studying γ , which roughly maps p to $L - mp$, we look at the map sending p to $p - p'$, where $p' \neq p$ is another point of intersection between E_p and \mathcal{C}_t .

More precisely, we let \mathcal{T} be the product $\mathcal{C} \times_{\mathbb{P}^1 \times \Delta} \mathcal{C}$ with diagonal removed. We have a well-defined map

$$(2.53) \quad \xi : \mathcal{T} \rightarrow \overline{\mathcal{M}}_{1,2}$$

sending $(p, p') \in \mathcal{T}$ to $(J(E_p), 0, p - p')$. Clearly, γ is dominant if ξ is; ξ is dominant if

$$(2.54) \quad \dim(\xi(\mathcal{T}) \cap \overline{\mathcal{M}}_{0,4}) = 1.$$

As $t \rightarrow 0$, the fibers $E_p \in |F|$ of $\pi : \mathcal{S}_t \rightarrow \mathbb{P}^1$ will degenerate to a curve $N \in |F|$ on the central fiber $\mathcal{S}_0 = R_1 \cup R_2$ as described in Remark 2.7. That is, N is a union $N_1 \cup N_2 \cup \dots \cup N_m$ given by (2.37). For N a general member of the pencil $|F|$, N_1 meets C_1 transversely at two distinct points $p \neq p' \notin D$, where $\mathcal{C}_0 = C_1 \cup C_2$ is the limiting rational curve constructed in the proof of Lemma 2.8. Clearly, $(p, p') \in \mathcal{T}$. It is not hard to see that ξ simply sends (p, p') to $(p, p', q, q') \in \overline{\mathcal{M}}_{0,4}$ as four points on $N_1 \cong \mathbb{P}^1$, where $N_1 \cap D = \{q, q'\}$. To show (2.54), it suffices to show that the moduli of (p, p', q, q') varies when N moves in the pencil $|F|$. From the construction of $C_1 \cup C_2$, we see that C_1 has a node $r \notin D$. For N a general member of $|F|$, (p, p', q, q') are four distinct points on N_1 . When N_1 passes through r , we have $p = p' = r$ while $r \neq q$ and $r \neq q'$. So the moduli of (p, p', q, q') changes as N varies. We are done. \square

2.4. Proof of Proposition 2.4. By Lemma 2.8, we can find an irreducible nodal rational curve $B \in |nL - C|$ for n sufficiently large. It also follows that there is an irreducible nodal elliptic curve $B \in |nL - C|$. It remains to show that we can deform $B \cup C$ to a rational curve if $g(B) = 0$ or an elliptic curve if $g(B) = 1$ on a general $K3$ surface.

Let us do the case $g(B) = 0$. We fix an intersection $p \in B \cap C$. Let B^ν and C^ν be the normalizations of B and C , respectively, and let

$$(2.55) \quad \eta : B^\nu \vee_p C^\nu \rightarrow B \cup C \subset X$$

be a partial normalization of $B \cup C$, where $B^\nu \vee_p C^\nu$ is the union of B^ν and C^ν meeting transversely at a single point over p . We want to show that the stable map $\eta : B^\nu \vee_p C^\nu \rightarrow X$ can be deformed when we deform X . This can be done by computing the virtual dimension of the moduli space of stable

maps to $K3$ surfaces. However, it cannot be done in a naive way for the following well-known reason: the virtual dimension of $\mathcal{M}_{nL,0}^X$ is

$$(2.56) \quad c_1(T_X) \cdot nL + \dim X - 3 = -1$$

which is not the “expected” dimension 0, where $\mathcal{M}_{\gamma,g}^X$ is the moduli space of stable maps $\eta : A \rightarrow X$ of genus g with $[\eta_*A] = \gamma \in H_2(X, \mathbb{Z})$. The remedy for this situation is to replace X by a so-called “twisted” family of $K3$ surfaces, i.e., a complex deformation of X . This way we have the right dimension and the stable maps $\eta \in \mathcal{M}_{nL,0}^X$ only deform onto projective $K3$ surfaces.

Let \mathcal{S}/Δ^2 be a family of complex $K3$ surfaces over the 2-disk Δ^2 with $\mathcal{S}_0 = X$ and the class $L \in H_2(\mathcal{S}, \mathbb{Z})$. Then $\eta \in \mathcal{M}_{nL,0}^{\mathcal{S}}$ and the virtual dimension of $\mathcal{M}_{nL,0}^{\mathcal{S}}$ is

$$(2.57) \quad c_1(T_{\mathcal{S}}) \cdot nL + \dim \mathcal{S} - 3 = 1.$$

Let V be an irreducible component of $\mathcal{M}_{nL,0}^{\mathcal{S}}$ containing η . Then $\dim V \geq 1$. Let

$$(2.58) \quad W = \{t \in \Delta^2 : L \in \text{Pic}(\mathcal{S}_t)\}$$

be the subvariety of Δ^2 parameterizing projective $K3$ surfaces polarized by L . Obviously, $\dim W = 1$ and $\text{Pic}(\mathcal{S}_t) = \mathbb{Z}$ for $t \neq 0 \in W$ and a general choice of \mathcal{S} . Clearly, V maps to W under the projection $\mathcal{S} \rightarrow \Delta^2$ and it is obviously flat over W since $\dim V_0 = 0$ and $\dim V \geq \dim W$. Hence $V_t \neq \emptyset$ for $t \neq 0 \in W$.

The case $g(B) = 1$ follows from the same argument. This finishes the proof of Proposition 2.4 and hence Theorem 1.1 follows. We need to say a few things more for Theorem 1.7.

By deformation theory, B moves in a one-parameter family when $g(B) = 1$. A general member of this family is an irreducible nodal elliptic curve meeting C transversely. In addition, the intersection $p \in B \cap C$ moves on C when B varies in the family. Now we let \mathcal{S} be a projective family of $K3$ surfaces polarized by L over Δ with $\mathcal{S}_0 = X$. We can deform $B \cup C$ to an irreducible elliptic curve on a general fiber \mathcal{S}_t of the family \mathcal{S}/Δ by the argument above. Namely, there exists a family of curves $\mathcal{C} \subset \mathcal{S}$, after a base change, such that $\mathcal{C}_0 = B \cup C$ and $g(\mathcal{C}_t) = 1$ for $t \neq 0$. Let $\nu : \mathcal{C}^\nu \rightarrow \mathcal{C}$ be the normalization of \mathcal{C} . Then \mathcal{C}_0^ν is the union $B^\nu \vee_p C^\nu$ described above. We lift $\nu : \mathcal{C}^\nu \rightarrow \mathcal{S}$ to $d\nu : \mathcal{C}^\nu \dashrightarrow \mathbb{P}T_{\mathcal{S}/\Delta}$. Let $\mu : \tilde{\mathcal{C}} \rightarrow \mathbb{P}T_{\mathcal{S}/\Delta}$ be the stable reduction of the map $d\nu$ and let \tilde{B} and $\tilde{C} \subset \tilde{\mathcal{C}}_0$ be the proper transforms of B and C , respectively.

Since B and C meet transversely at p , the images of the tangent spaces $T_{B,p}$ and $T_{C,p}$ in $T_{X,p}$ differ. Consequently, $\mu(\tilde{B})$ and $\mu(\tilde{C})$ meet $\mathbb{P}T_{X,p}$ at two distinct points, where $\mathbb{P}T_{X,p} \cong \mathbb{P}^1$ is the fiber of $\mathbb{P}T_{\mathcal{S}/\Delta}/\mathcal{S}$ over the point $p \in \mathcal{S}$. Therefore, \tilde{B} and \tilde{C} are disjoint on $\tilde{\mathcal{C}}_0$ and they must be joined by a tree of rational curves that dominates $\mathbb{P}T_{X,p}$. That is, $\mathbb{P}T_{X,p} \subset \mu(\tilde{\mathcal{C}})$ and

hence $\mathbb{P}T_{X,p} \subset \varphi(\mathcal{E}_{g,n})$. As p moves on C , we see that

$$(2.59) \quad \bigcup_{p \in C} \mathbb{P}T_{X,p} \subset \overline{\varphi(\mathcal{E}_{g,n})}.$$

We take C to be a member of a sequence of rational curves which are dense on X . Hence

$$(2.60) \quad \mathbb{P}T_X \subset \bigcup_{n=1}^{\infty} \overline{\varphi(\mathcal{E}_{g,n})}$$

and Theorem 1.7 follows.

2.5. The case $g = 2$. A $K3$ surface in $\mathcal{P}_{2,m}$ can still be degenerated to a union $R_1 \cup R_2$ with $R_i \cong \mathbb{F}_1$ and L_i and F_i given by (2.31), (2.33) and (2.34), just as in the case that $g \geq 4$ is even. Let \mathcal{S}/Δ be the corresponding family of $K3$ surfaces with $\mathcal{S}_0 = R_1 \cup R_2$ and $(\mathcal{S}_t, L) \in \mathcal{P}_{2,m}$. Such \mathcal{S} is projective over Δ since $L + nF$ is relatively ample over Δ for all $n > 0$. However, L is big and nef but not ample over Δ itself. Indeed, the birational map $\psi : \mathcal{S} \rightarrow \mathcal{Q}$ given by $|nL|$ for $n \geq 2$ contracts the two exceptional curves M_i . The 3-fold \mathcal{Q} is a family of $K3$ surfaces in $\mathcal{P}_{2,m}$ over Δ whose central fiber $\mathcal{Q}_0 = S_1 \cup S_2$ is a union of $S_i \cong \mathbb{P}^2$ meeting transversely along an elliptic curve $D = S_1 \cap S_2$. Here we use the same notation D for both intersections $R_1 \cap R_2$ and $S_1 \cap S_2$.

The two curves M_i are contracted by ψ to two rational double points p_{17} and p_{18} of \mathcal{Q} on $D = S_1 \cap S_2$. Indeed, \mathcal{Q} has eighteen rational double points $p_1, p_2, \dots, p_{16}, p_{17}, p_{18}$ on D by deformation theory, where p_1, p_2, \dots, p_{16} are the images of the rational double points of \mathcal{S} under ψ . Again we use the same notations p_1, p_2, \dots, p_{16} for both the rational double points of \mathcal{S} and their images under ψ .

One subtle point is that M_i are contracted to the same point $p_{17} = p_{18}$ where \mathcal{Q} has a singularity of the type $xy = tz^2$ when $m = 2$. Such a singularity can be analyzed in the same way as rational double points. Basically, we have two rational double points “collide” in this special case. However, we can save ourselves some trouble in dealing with this “corner” case by simply assuming that $m \geq 4$ since

$$(2.61) \quad \bigcup_{\substack{2|m \\ m \geq m_0}} \mathcal{P}_{g,m}$$

is obviously dense in \mathcal{K}_g for all m_0 . For our purpose, we may simply assume m to be sufficiently large. So p_{17} and p_{18} are two distinct points on D and R_i is the blowup of S_i at p_{16+i} for $i = 1, 2$, respectively. And $\psi : \mathcal{S} \rightarrow \mathcal{Q}$ is a small resolution of \mathcal{Q} at p_{17} and p_{18} . It is well known that there are flops

of \mathcal{S} with respect to M_i . Namely, we have the diagram

$$(2.62) \quad \begin{array}{ccc} \mathcal{S} & \overset{\dashrightarrow}{\text{-----}} & \mathcal{S}' \\ & \searrow \psi & \swarrow \psi' \\ & \mathcal{Q} & \end{array}$$

where \mathcal{S}' is the 3-fold obtained from \mathcal{S} by flops with respect to M_1 and M_2 . That is, the central fiber $\mathcal{S}'_0 = R'_1 \cup R'_2$ of \mathcal{S}' is a union of $R'_i \cong \mathbb{F}_1$ with R'_i the blowup of S_i at p_{19-i} for $i = 1, 2$.

Let L' and $F' \in \text{Pic}(\mathcal{S}'/\Delta)$ be the proper transforms of L and F , respectively, and let M'_i and G'_i be the generators of $\text{Pic}(R'_i)$ given in the same way as (2.32). It is not hard to see that

$$(2.63) \quad L'_i = L' \Big|_{R'_i} = M'_i + G'_i \text{ and } F'_i = F' \Big|_{R'_i} = m M'_i + \frac{m}{2} G'_i.$$

So we can work with either \mathcal{S} or \mathcal{S}' to produce rational curves in $|aL + bF|$ on \mathcal{S}_t or equivalently $|aL' + bF'|$ on \mathcal{S}'_t , depending on the sign of b .

Proof of Lemma 2.8 when $g = 2$. When $b > 0$, we have

$$(2.64) \quad (aL + bF) \Big|_{R_i} = aM_i + \left(a + \frac{bm}{2}\right) G_i$$

with $a + bm/2 > a$. Hence we may use the same construction of limiting rational curves $C_1 \cup C_2$ as in the case of g being even and $g \geq 4$.

When $b < 0$, we have

$$(2.65) \quad (aL' + bF') \Big|_{R'_i} = (a + bm)M'_i + \left(a + \frac{bm}{2}\right) G'_i$$

where $a + bm > 0$ by (2.38) and $a + bm/2 > a + bm$. So we may use the same construction again by working with \mathcal{S}' . \square

The proof of Lemma 2.6 goes through without any change since we are using the limiting rational curves in $|2L + F|$ for which no flops are needed.

3. NORMAL FUNCTIONS ASSOCIATED TO ELLIPTIC FIBRATIONS

Here we will give another proof of Proposition 2.3 via the theory of normal functions. Roughly, we will show that if $L - mp$ fails to generate a dense subgroup of $J(X_q)$ for a general point $q \in \mathbb{P}^1$, then it has to be torsion for all q . The advantage of this approach is that it does not seem to depend on the general moduli of X , although we do need the fact, which we will prove by degeneration, that the rational curve $C \subset X$ we start with meets the singular fibers of X/\mathbb{P}^1 transversely.

Given an elliptic surface $X_\Gamma \rightarrow \Gamma$, we let $\Sigma \subset \Gamma$ correspond to the singular fibers of $\rho_\Gamma : X_\Gamma \rightarrow \Gamma$, with inclusion $j : U := \Gamma \setminus \Sigma \hookrightarrow \Gamma$. So we have a

diagram:

$$\begin{array}{ccc} X_U & \hookrightarrow & X_\Gamma \\ \rho_U \downarrow & & \downarrow \rho_\Gamma \\ U & \xrightarrow{j} & \Gamma, \end{array}$$

where ρ_U is smooth and proper. The local invariant cycle property (see [Z2], §15) gives us a surjection:

$$R^i \rho_{\Gamma,*} \mathbb{C} \rightarrow j_* R^i \rho_{U,*} \mathbb{C},$$

for all i and hence

$$H^1(\Gamma, R^1 \rho_{\Gamma,*} \mathbb{C}) \simeq H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{C}).$$

The Leray spectral sequence for ρ_Γ degenerates at E_2 (see [Z2], §15). This is induced by a Leray filtration: $H^2(X_\Gamma, \mathbb{Q}) =$

$$L^0 H^2(X, \mathbb{Q}) \supset L^1 H^2(X_\Gamma, \mathbb{Q}) \supset L^2 H^2(X_\Gamma, \mathbb{Q}) \supset L^3 H^2(X_\Gamma, \mathbb{Q}) = 0.$$

Let $Gr_L^i H^2(X_\Gamma, \mathbb{Q}) = L^i H^2(X_\Gamma, \mathbb{Q}) / L^{i+1} H^2(X_\Gamma, \mathbb{Q})$. Note that

$$Gr_L^2 H^2(X_\Gamma, \mathbb{Q}) = L^2 H^2(X_\Gamma, \mathbb{Q}) = H^2(\Gamma, R^0 \rho_{\Gamma,*} \mathbb{Q}) = \mathbb{Q}[F] \simeq \mathbb{Q},$$

where we use the fact that $R^0 \rho_{\Gamma,*} \mathbb{Q} \simeq \mathbb{Q}$ is the constant sheaf. Further,

$$Gr_L^1 H^2(X_\Gamma, \mathbb{Q}) = H^1(\Gamma, R^1 \rho_{\Gamma,*} \mathbb{Q}) \simeq H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Q}),$$

and the kernel of the surjective map

$$H^2(X_\Gamma, \mathbb{Q}) \rightarrow Gr_L^0 H^2(X_\Gamma, \mathbb{Q}) = H^0(\Gamma, R^2 \rho_{\Gamma,*} \mathbb{Q}),$$

defines $L^1 H^2(X_\Gamma, \mathbb{Q})$. There are short exact sequences:

$$0 \rightarrow \mathbb{Q}[F] \rightarrow L^1 H^2(X_\Gamma, \mathbb{Q}) \rightarrow H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Q}) \rightarrow 0,$$

$$0 \rightarrow H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Q}) \rightarrow \frac{H^2(X_\Gamma, \mathbb{Q})}{\mathbb{Q}[F]} \rightarrow H^0(\Gamma, R^2 \rho_{\Gamma,*} \mathbb{Q}) \rightarrow 0.$$

There is a commutative diagram

$$(3.1) \quad \begin{array}{ccccccc} 0 & \rightarrow & H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Q}) & \rightarrow & \frac{H^2(X_\Gamma, \mathbb{Q})}{\mathbb{Q}[F]} & \rightarrow & H^0(\Gamma, R^2 \rho_{\Gamma,*} \mathbb{Q}) \rightarrow 0 \\ & & \rho_{\Gamma,*} \downarrow & & \downarrow \bar{\rho}_{\Gamma,*} & & \\ & & H^0(\Gamma, \mathbb{Q}) & = & H^0(\Gamma, \mathbb{Q}), & & \end{array}$$

where $\bar{\rho}_{\Gamma,*}$ is induced from $\rho_{\Gamma,*}$. Note that $\ker(\bar{\rho}_{\Gamma,*})$ will involve the components of the bad fibers of ρ_Γ . Let $F_t := \rho_\Gamma^{-1}(t)$. There are holomorphic

vector bundles over U :

$$\mathcal{F}^1 := \mathcal{O}_\Gamma \left(\prod_{t \in U} H^{1,0}(F_t, \mathbb{C}) \right) \subset \mathcal{F} := \mathcal{O}_\Gamma \left(\prod_{t \in U} H^1(F_t, \mathbb{C}) \right), \quad \mathcal{F}^{1,*} := \mathcal{F}/\mathcal{F}^1,$$

with canonical extensions

$$\overline{\mathcal{F}}^1 \subset \overline{\mathcal{F}}, \quad \overline{\mathcal{F}}^{1,*} := \overline{\mathcal{F}}/\overline{\mathcal{F}}^1,$$

over Γ (see [Z2], §3), as well as a short exact sequences of sheaves:

$$0 \rightarrow R^1 \rho_{U,*} \mathbb{Z} \rightarrow \mathcal{F}^{1,*} \rightarrow \mathcal{J} \rightarrow 0 \quad (\text{over } U),$$

$$0 \rightarrow j_* R^1 \rho_{U,*} \mathbb{Z} \rightarrow \overline{\mathcal{F}}^{1,*} \rightarrow \overline{\mathcal{J}} \rightarrow 0 \quad (\text{over } \Gamma),$$

where \mathcal{J} , $\overline{\mathcal{J}}$ are the sheaves of germs of normal functions over U and Γ respectively. From the work of [Z2], $H^i(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Z})$ is naturally endowed with a pure Hodge structure of weight $i+1$; moreover from ([Z2], §9), one has isomorphisms:

$$H^1(\Gamma, \overline{\mathcal{F}}^{1,*}) = \frac{H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{C})}{F^1 H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{C})},$$

$$H^0(\Gamma, \overline{\mathcal{F}}^{1,*}) = \frac{H^0(\Gamma, R^1 \rho_{U,*} \mathbb{C})}{F^1 H^0(\Gamma, R^1 \rho_{U,*} \mathbb{C})}.$$

(It is worthwhile pointing out that outside of cases of trivial j -invariant, one has $H^0(\Gamma, R^1 \rho_{U,*} \mathbb{C}) = 0$ (see [C-Z], p. 5).) Taking cohomology, one has a short exact sequence:

$$0 \rightarrow \frac{H^0(\Gamma, \overline{\mathcal{F}}^{1,*})}{H^0(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Z})} \rightarrow H^0(\Gamma, \overline{\mathcal{J}}) \xrightarrow{\delta} H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Z})^{1,1} \rightarrow 0,$$

where $H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Z})^{1,1} :=$

$$\ker \left(H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Z}) \rightarrow H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{C}) / F^1 H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{C}) \right).$$

The group $H^0(\Gamma, \overline{\mathcal{J}})$ is called the group of normal functions, and for $\nu \in H^0(\Gamma, \overline{\mathcal{J}})$, $\delta(\nu) \in H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Z})$ is called its topological invariant. We need the following key observation:

Proposition 3.1. *Suppose that $\nu \in H^0(\Gamma, \overline{\mathcal{J}})$ is given such that $\delta(\nu)$ is nontorsion. Then for sufficiently general $t \in U$, the cyclic group generated by $\nu(t)$ is dense in $J^1(E_t)$.*

Proof. A local lifting of the normal function $\nu|_U \in H^0(U, \mathcal{J})$ determines an analytic function on a disk $\Delta \subset U$, viz., $\tilde{\nu} \in H^0(\Delta, \mathcal{F}^{1,*}) \simeq H^0(\Delta, \mathcal{O}_\Delta)$, using the fact that $\mathcal{F}^{1,*}$ is a holomorphic line bundle. Further, we have the family of lattices $H^0(\Delta, R^1 \rho_{U,*} \mathbb{Z}) \hookrightarrow H^0(\Delta, \mathcal{F}^{1,*})$. Let $\delta_1, \delta_2 \in H^0(\Delta, R^1 \rho_{U,*} \mathbb{Z})$ be generators with respective images $[\delta_1], [\delta_2] \in H^0(\Delta, \mathcal{F}^{1,*})$, under the (injective) composite

$$H^0(\Delta, R^1 \rho_{U,*} \mathbb{Z}) \rightarrow H^0(\Delta, \mathcal{F}) \rightarrow H^0(\Delta, \mathcal{F}^{1,*}).$$

Thus we can write

$$\tilde{\nu}(t) = x(t)[\delta_{1,t}] + y(t)[\delta_{2,t}],$$

for unique real-valued functions $x(t)$, $y(t)$, $t \in \Delta$. Note that $[\delta_{2,t}] = g(t)[\delta_{1,t}]$ for some holomorphic function $g(t)$, and likewise $\tilde{\nu}(t) = h(t)[\delta_{1,t}]$ for a holomorphic $h(t)$. Thus $h(t) = x(t) + y(t)g(t)$, and in particular $\operatorname{Re}(h(t)) = x(t) + y(t)\operatorname{Re}(g(t))$, $\operatorname{Im}(h(t)) = y(t)\operatorname{Im}(g(t))$. Thus $x(t)$ and $y(t)$ are real analytic functions. If the cyclic group generated by $\nu(t)$ is not dense in $J^1(F_t)$ for uncountably many $t \in \Delta$, then by a countability and Baire type argument together with Lemma 2.5, $\{1, x(t), y(t)\}$ lie on a hyperplane $a_1x + a_2y + a_3 = 0$ in \mathbb{R}^2 , where $\{a_j\} \in \mathbb{Q}$ are constant with respect to $t \in \Delta$ and not all zero. Using $h(t) = x(t) + y(t)g(t)$, one can easily check then that $a_1x(t) + a_2y(t) + a_3 = 0$ for all $t \in \Delta$ implies that $\tilde{\nu}$ is constant. More precisely, one can choose the lift $\tilde{\tilde{\nu}} \in H^0(\Delta, R^1\rho_{U,*}\mathbb{C})$ of $\tilde{\nu}$ via the composite

$$H^0(\Delta, R^1\rho_{\Delta,*}\mathbb{C}) \rightarrow H^0(\Delta, \mathcal{F}) \rightarrow H^0(\Delta, \mathcal{F}/\mathcal{F}^1) = H^0(\Delta, \mathcal{F}^{1,*}), \quad \tilde{\tilde{\nu}} \mapsto \tilde{\nu}.$$

This tells us that the Griffiths' infinitesimal invariant of ν over U (see [G], p. 69) is zero. However in this case the Griffiths' infinitesimal invariant is known to coincide with the topological de Rham invariant (see [MS], as well as [L-S] for a background on this). In the end, this translates to saying that $\delta(\nu|_U) = 0$ as a class in $H^1(U, R^1\rho_{U,*}\mathbb{Q})$. Alternatively and more directly, if we assume for the moment that the j -invariant of the family $X_U \rightarrow U$ is nonconstant, then $\mathcal{F}^1 \cap R^1\rho_{U,*}\mathbb{C} = 0 \in \mathcal{F}$ and hence $\nu|_U$ is induced by a class in $H^0(U, R^1\rho_{U,*}\mathbb{C}/R^1\rho_{U,*}\mathbb{Z})$, and therefore $\delta(\nu|_U) = 0 \in H^1(U, R^1\rho_{U,*}\mathbb{Q})$. The same conclusion holds, albeit a tedious argument, if the j -invariant is constant - the details are left to the reader and involve a generalization of Example 3.2 below. Next by ([Z2], §14), the map $H^1(\Gamma, R^1\rho_{\Gamma,*}\mathbb{Q}) \hookrightarrow H^1(U, R^1\rho_{U,*}\mathbb{Q})$ is injective. Hence $\delta(\nu) = 0 \in H^1(\Gamma, R^1\rho_{\Gamma,*}\mathbb{Q})$, a contradiction. \square

Example 3.2. Let E be an elliptic curve and $Y = E \times E$. We can illustrate Proposition 3.1 rather easily in this situation. With regard to the first projection $Y \rightarrow E$, the sheaf of germs of normal functions \mathcal{J} is given by the short exact sequences of sheaves over E :

$$0 \rightarrow H^1(E, \mathbb{Z}) \rightarrow \mathcal{O}_E(H^{0,1}(E)) \rightarrow \mathcal{J} \rightarrow 0.$$

Note that $H^1(E, \mathcal{O}_E(H^{0,1}(E))) \simeq H^{0,1}(E) \otimes H^{0,1}(E)$, and hence there is the short exact sequence:

$$0 \rightarrow J^1(E) \rightarrow H^0(E, \mathcal{J}) \xrightarrow{\delta} \{H^1(E, \mathbb{Z}) \otimes H^1(E, \mathbb{Z})\} \bigcap H^{1,1}(Y) \rightarrow 0.$$

If $\nu \in H^0(E, \mathcal{J})$ has trivial infinitesimal invariant, then

$$\nu \in H^0(E, H^{0,1}(E)/H^1(E, \mathbb{Z})) \simeq J^1(E),$$

and hence $\delta(\nu) = 0$. For $n \in \mathbb{N}$, let $f_n : E \rightarrow E$ be given by multiplication by n , and let $\Xi(n)$ be the graph of f_n in Y , with Künneth component

$[\Xi(n)^{1,1}] \in H^1(E, \mathbb{Z}) \otimes H^1(E, \mathbb{Z})$. It follows rather directly from Lemma 2.5 that

$$\bigcup_{n \in \mathbb{N}} \Xi(n) \subset Y,$$

is dense in Y in the strong topology. Note however that if ν is the normal function associated to f_1 , then $n\nu$ is the normal function associated to f_n . Furthermore $\delta(n\nu) = [\Xi(n)^{1,1}] \neq 0$, and hence the density also follows from Proposition 3.1. Now let $X := Y/\pm$ be the corresponding Kummer counterpart with C_n being the image of $\Xi(n)$ in X . Then C_n is a rational curve and

$$\bigcup_{n \in \mathbb{N}} C_n \subset X,$$

is likewise dense in X in the strong topology.

This, combining with Theorem 1.5 of Bogomolov and Tschinkel, gives us the density of rational curves in the strong topology on every elliptic $K3$ surface X with $\text{rank Pic}(X) \leq 19$, i.e., Theorem 1.6. It also gives another proof of Proposition 2.3 as long as we can find a rational non torsion multisection, which is a consequence of Theorem 1.5. On the other hand, we can produce such a multisection using our deformational argument as follows. Although this is redundant, we keep it here to make our paper self-contained.

Now let us consider a general elliptic $K3$ surface $(X, L) \in \mathcal{P}_{g,m}$ with $\rho : X \rightarrow \mathbb{P}^1$ the elliptic fibration given by $|F|$.

Let $\Gamma_0 \in |L|$ be a rational curve, with desingularization $\Gamma = \mathbb{P}^1$. Note that $\rho|_{\Gamma_0} : \Gamma_0 \rightarrow \mathbb{P}^1$ has degree m , and hence the corresponding map $\lambda : \Gamma \rightarrow \mathbb{P}^1$ is of degree m . Base change gives us an elliptic surface $\rho_\Gamma : X_\Gamma \rightarrow \Gamma$, with section $\sigma : \Gamma \hookrightarrow X_\Gamma$, (where we can assume after a proper modification, that X_Γ is smooth). Let $h : X_\Gamma \rightarrow X$ be the obvious morphism (of degree m). Note that

$$h_*(\sigma(\Gamma)) = \Gamma_0;$$

moreover we have corresponding classes $F, h^*(L)$ on X_Γ , with $h_*(F) = F$, and on X_Γ :

$$F^2 = 0, (h^*(L))^2 = m \cdot (2g - 2), (\sigma(\Gamma))^2 = b, (\text{some } b \in \mathbb{Z}),$$

$$F \cdot \sigma(\Gamma) = 1, F \cdot h^*(L) = m, \sigma(\Gamma) \cdot h^*(L) = 2g - 2.$$

Note that $\{F, h^*(L), \sigma(\Gamma)\}$ are independent over \mathbb{Q} iff $m \cdot b \neq 2g - 2$. The independence follows from

Lemma 3.3. $b < 0$.

Proof. Since $\sigma(\Gamma) = \mathbb{P}^1$, the adjunction formula tells us that

$$-2 = b + K_{X_\Gamma} \cdot \sigma(\Gamma).$$

But h is ramified only along the fibers of ρ_Γ , i.e. over which λ ramifies, and hence

$$K_{X_\Gamma} = h^*(K_X) + k \cdot F,$$

for some integer $k \geq 0$. But X a K3 surface implies that $K_X = 0$, and hence $b = -(2 + k) < 0$. \square

Now let us suppose that:

$$(3.2) \quad \bar{\rho}_{\Gamma,*} \text{ in diagram (3.1) is an isomorphism.}$$

Then using the fact that $\rho_{\Gamma,*}(h^*(L)) = m\Gamma$, by (3.1) it follows that

$$[h^*(L) - m\sigma(\Gamma)] \neq 0 \in H^1(\Gamma, j_*R^1\rho_{\Gamma,*}\mathbb{Z})^{1,1}.$$

The general story (viz., when (3.2) is not satisfied) involves a rational linear combination of the components of the bad fibers of ρ_{Γ} together with $[h^*(L) - m\sigma(\Gamma)]$ (compare for example ([C-Z], thm 1.6)). The argument in showing that this gives a nontrivial class in $H^1(\Gamma, j_*R^1\rho_{\Gamma,*}\mathbb{Z})^{1,1}$ is similar but more complicated. For our purpose, we can always choose Γ_0 such that it meets the singular fibers of X/\mathbb{P}^1 transversely and hence X_{Γ} is smooth and has irreducible fibers over Γ and (3.2) is trivially satisfied. Then $[h^*(L) - m\sigma(\Gamma)]$ determines a normal function ν with

$$\delta(\nu) = [h^*(L) - m\sigma(\Gamma)] \in H^1(\Gamma, j_*R^1\rho_{\Gamma,*}\mathbb{Z})^{1,1}$$

and hence by Proposition 3.1, $\nu(t)$ has nontrivial dynamics for general $t \in \Gamma$.

It remains to verify the following.

Lemma 3.4. *For all $g, m \in \mathbb{Z}^+$ satisfying $g \geq 2$ and $2|m$ and a general $(X, L) \in \mathcal{P}_{g,m}$, there is an irreducible nodal rational curve in $|L|$ that meets all singular curves in $|F|$ transversely.*

Proof. It is well known that X/\mathbb{P}^1 has 24 nodal fibers. It suffices to figure out where these 24 curves in $|F|$ go when we degenerate X . Let \mathcal{S}/Δ be the family of K3 surfaces constructed in 2.3. A curve $N \in |F|$ on the central fiber $\mathcal{S}_0 = R = R_1 \cup R_2$ is described in Remark 2.7. It is not hard to see that N is a limit of nodal rational curves in $|F|$ on the general fibers if one of the following holds:

- N passes through one of the sixteen rational double points p_j and there are sixteen such curves;
- N passes through one of the four points $\{p \in D : G_1 \sim_{\text{rat}} 2p \text{ on } D\}$ and there are four such curves;
- N passes through one of the four points $\{p \in D : G_2 \sim_{\text{rat}} 2p \text{ on } D\}$ and there are four such curves.

One can check that these add up to 24.

Now we let $C = C_1 \cup C_2$ with irreducible components $C_i \in |L_i|$ satisfying

$$(3.3) \quad C_1 \cdot D = C_2 \cdot D = (g + 1)q$$

for some point $q \in D$. This is a limiting rational curve and it obviously meets each of the 24 curves $N \in |F|$ given above transversely. \square

Remark 3.5. It is well known that for an elliptic curve E defined over an algebraically closed subfield $k \subset \mathbb{C}$, the torion subgroup $E_{\text{tor}}(\mathbb{C}) \subset E(k)$. An analogous result holds for rational curves on a K3 surface. Quite generally, the following result which may be common knowledge among experts, seems worthwhile mentioning:

Proposition 3.6. *Assume given X/\mathbb{C} a smooth projective surface with $\text{Pg}(X) := \dim H^{2,0}(X) > 0$. If we write $X/\mathbb{C} = X_k \times_k \mathbb{C}$, viz., X/\mathbb{C} obtained by base change from a smooth projective surface X_k defined over an algebraically closed subfield $k \subset \mathbb{C}$, and if $C \subset X/\mathbb{C}$ is a rational curve, then C is likewise defined over k .*

Proof. By a standard spread argument, there is a smooth projective variety S/k of dimension ≥ 0 , and a k -family $\mathcal{C} \rightarrow S$ of rational curves containing C as a very general member, with embedding h :

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{h} & S \times_k X & \xrightarrow{Pr_X} & X \\ Pr_S \searrow & & \swarrow & & \\ & & S & & \end{array}$$

Since $\text{Pg}(X) > 0$, there are only at most a countable number of rational curves on X/\mathbb{C} , and hence $Pr_X(h(\mathcal{C})) = Pr_X(h(Pr_S^{-1}(t)))$ for any $t \in S(\mathbb{C})$. Now use the fact that $S(k) \neq \emptyset$. \square

4. APPENDIX

It seems appropriate to provide one possible generalization of Proposition 3.1. A good reference for the background material required here is [Z1]. Let X/\mathbb{C} be a projective algebraic manifold of dimension $2m$ and $\{X_t\}_{t \in \mathbb{P}^1}$ a Lefschetz pencil of hyperplane sections of X . Let $D := \bigcap_{t \in \mathbb{P}^1} X_t$ be the (smooth) base locus and $\bar{X} = B_D(X)$, the blow-up. One has a diagram:

$$(4.1) \quad \begin{array}{ccc} \bar{X}_U & \hookrightarrow & \bar{X} \\ \rho_U \downarrow & & \downarrow \rho \\ U & \xrightarrow{j} & \mathbb{P}^1, \end{array}$$

where $\Sigma := \mathbb{P}^1 \setminus U = \{t_1, \dots, t_M\}$ is the singular set, viz., where the fibers are singular Lefschetz hyperplane sections. One has a short exact sequence of sheaves

$$(4.2) \quad 0 \rightarrow j_* R^{2m-1} \rho_* \mathbb{Z} \rightarrow \bar{\mathcal{F}}^{m,*} \rightarrow \bar{\mathcal{J}} \rightarrow 0,$$

where

$$\bar{\mathcal{F}}^{m,*} = \mathcal{O}_{\mathbb{P}^1} \left(\prod_{t \in \mathbb{P}^1} \frac{H^{2m-1}(X_t, \mathbb{C})}{F^m H^{2m-1}(X_t, \mathbb{C})} \right) \quad (\text{canonical extension}),$$

and where $\overline{\mathcal{J}}$ is the sheaf of germs of normal functions. The results in ([Z1], Cor. 4.52) show that (4.2) induces a short exact sequence:

$$(4.3) \quad 0 \rightarrow J^m(X) \rightarrow H^0(\mathbb{P}^1, \overline{\mathcal{J}}) \xrightarrow{\delta} H^1(\mathbb{P}^1, R^{2m-1}\rho_{U,*}\mathbb{Z})^{(m,m)} \rightarrow 0,$$

where $H^1(\mathbb{P}^1, R^{2m-1}\rho_{U,*}\mathbb{Z})^{(m,m)}$ are the integral classes of Hodge type (m, m) in $H^1(\mathbb{P}^1, R^{2m-1}\rho_{U,*}\mathbb{Z})$, and the fixed part $J^m(X)$ is the Griffiths intermediate jacobian. For $t \in U$, the Lefschetz theory guarantees an orthogonal decomposition

$$H^{2m-1}(X_t, \mathbb{Z}) = H^{2m-1}(X, \mathbb{Z}) \oplus H_v^{2m-1}(X_t, \mathbb{Z}),$$

where by the weak Lefschetz theorem, $H^{2m-1}(X, \mathbb{Z})$ is identified with its image $H^{2m-1}(X, \mathbb{Z}) \hookrightarrow H^{2m-1}(X_t, \mathbb{Z})$ and $H_v^{2m-1}(X_t, \mathbb{Z})$ is the space generated by the vanishing cocycles $\{\delta_1, \dots, \delta_M\}$. Note that a basis for $H_v^{2m-1}(X_t, \mathbb{Z})$ is given (up to relabelling) by a suitable subset $\{\delta_1, \dots, \delta_{2g}\}$ of vanishing cocycles. We are going to make the following ‘‘artificial’’ assumption:

$$(4.4) \quad \prod_{j=1}^{2g-1} (\delta_j, \delta_{j+1})_{X_t} \neq 0,$$

where $(\delta_j, \delta_{j+1})_{X_t}$ is the cup product as an integer (i.e., followed by the trace). We do not need this assumption, except for the fact that it simplifies the proof of the following:

Theorem 4.1. *Let $\nu \in H^0(\mathbb{P}^1, \overline{\mathcal{J}})$ be given such that $\delta(\nu)$ is nontorsion. Then for very general $t \in U$, the subgroup $\langle \nu(t) \rangle \subset J^m(X_t)$ generated by $\nu(t)$, is dense in the strong topology. In particular, the family of rational curves in the manifold (see [Z1], Prop. 2.9):*

$$\mathbf{J} := \coprod_{t \in \mathbb{P}^1} J^m(X_t),$$

is dense in the strong topology.

Proof. For each $t_j \in \Sigma$, one has the Picard-Lefschetz transformation T_j , $T_j(\gamma) = \gamma + (-1)^m(\gamma, \delta_j)\delta_j$ (here (γ, δ_j) means $(\gamma, \delta_j)_{X_t}$) for which

$$N_j = \log T_j = (T_j - I), \text{ using } (T_j - I)^2 = 0.$$

Now let $\nu \in H^0(\mathbb{P}^1, \overline{\mathcal{J}})$ and $\omega \in H^0(\mathbb{P}^1, \overline{\mathcal{F}}^m)$ be given. Note that

$$\nu : \mathbb{P}^1 \rightarrow \mathbf{J},$$

defines a rational curve on \mathbf{J} . Next, the images

$$\{[\delta_1], \dots, [\delta_{2g}]\} \text{ in } F^{m,*}H_v^{2m-1}(X_t, \mathbb{C}) := H_v^{2m-1}(X_t, \mathbb{C})/F^m H_v^{2m-1}(X_t, \mathbb{C}),$$

define a lattice. In terms of this lattice and modulo the fixed part $J^m(X)$, a local lifting of ν is given by $\sum_{j=1}^{2g} x_j(t)[\delta_j]$, for suitable real-valued functions $\{x_j(t)\}$, multivalued on U . Let $T_j\nu(\omega(t))$ be the result of analytic

continuation of $\nu(\omega(t))$ counterclockwise in \mathbb{P}^1 about t_j and $N_j\nu(\omega(t)) = T_j\nu(\omega(t)) - \nu(\omega(t))$. About t_j , we pick up a period

$$N_i\nu(\omega(t)) = c_i \int_{\delta_i} \omega(t), \text{ for some } c_i \in \mathbb{Q},$$

dependent only on ν , where we identify δ_i with its corresponding homology vanishing cycle via Poincaré duality. Likewise in terms of the lattice description,

$$\begin{aligned} N_i\nu(\omega(t)) &= \sum_{j=1}^{2g} T_i(x_j(t)) \int_{\delta_j + (-1)^m (\delta_j, \delta_i) \delta_i} \omega(t) - \sum_{j=1}^{2g} x_j(t) \int_{\delta_j} \omega(t) \\ &= \sum_{j=1}^{2g} N_i(x_j(t)) \int_{\delta_j} \omega(t) + (-1)^m \left(\sum_{j=1}^{2g} T_i(x_j(t)) (\delta_j, \delta_i) \right) \cdot \int_{\delta_i} \omega(t). \end{aligned}$$

Thus

$$(4.5) \quad c_i = N_i(x_i(t)) + \sum_{j=1}^{2g} T_i(x_j(t)) (\delta_j, \delta_i),$$

and $N_i(x_j(t)) = 0$ for all $i \neq j$. Hence $T_i(x_j(t)) = x_j(t)$ for all $i \neq j$ and further, using $(\delta_i, \delta_i) = 0$, we can rewrite equation (4.5) as:

$$(4.6) \quad c_i = N_i(x_i(t)) + \sum_{j=1}^{2g} x_j(t) (\delta_j, \delta_i).$$

Note that if $N_i(x_i(t)) = 0$ for all i , then from the linear system in (4.6), $x_i(t) \in \mathbb{Q}$ for all i , and so $\delta(\nu) = 0 \in H^1(\mathbb{P}^1, j_* R^{2m-1} \rho_{U,*} \mathbb{Q})$. Now suppose that we have a nontrivial relation:

$$\sum_{j=1}^{2g} \lambda_j x_j(t) = \lambda_0, \text{ for some } \lambda_i \in \mathbb{Q}, \forall i, t \in U.$$

Then we have

$$\lambda_i N_i(x_i(t)) = \sum_{j=1}^{2g} \lambda_j N_i(x_j(t)) = 0.$$

So $\lambda_i \neq 0 \Rightarrow N_i(x_i(t)) = 0$. Since $(\lambda_1, \dots, \lambda_{2g}) \neq (0, \dots, 0)$ we can assume that $\lambda_1 \neq 0$ say. So for $i = 1$:

$$c_1 = (\delta_2, \delta_1) x_2(t) + (\delta_3, \delta_1) x_3(t) + \dots + (\delta_{2g}, \delta_1) x_{2g}(t),$$

and applying N_2 and (4.4) we arrive at

$$0 = N_2(c_1) = (\delta_2, \delta_1) N_2(x_2(t)) \Rightarrow N_2(x_2(t)) = 0.$$

Thus

$$c_2 = (\delta_1, \delta_2) x_1(t) + (\delta_3, \delta_2) x_3(t) + \dots + (\delta_{2g}, \delta_2) x_{2g}(t).$$

Applying N_3 and (4.4) we arrive at

$$0 = N_3(c_2) = (\delta_3, \delta_2) N_3(x_3(t)) \Rightarrow N_3(x_3(t)) = 0,$$

and so on. Finally, without assumption (4.4), and based on the fact that the vanishing (co-)cycles $\{\delta_1, \dots, \delta_M\}$ are all conjugate under the monodromy group action, a different nontrivial product combination of cup-products can be used to replace (4.4), with a similar argument as above. The theorem follows from this. \square

Corollary 4.2. *Let V be a general quintic threefold. Then the image of the Abel-Jacobi map $AJ : \mathrm{CH}_{\mathrm{hom}}^2(V) \rightarrow J^2(V)$ is a countable dense subset of $J^2(V)$.*

Proof. Let $X \subset \mathbb{P}^5$ be the Fermat quintic fourfold, and $\{X_t\}_{t \in \mathbb{P}^1}$ a Lefschetz pencil of hyperplane sections of X . We will assume the notation given in diagram (4.1). For the Fermat quintic, it is easy to check that $H^1(\mathbb{P}^1, R^3\rho_{U,*}\mathbb{Q})^{(2,2)} \neq 0$, so by the sequence in (4.3), there exists $\nu \in H^0(\mathbb{P}^1, \overline{\mathcal{J}})$ such that $\delta(\nu) \neq 0 \in H^1(\mathbb{P}^1, R^3\rho_{U,*}\mathbb{Q})$ (this being related to Griffiths' famous example ([Gr])). Thus by Theorem 4.1 and for general $t \in \mathbb{P}^1$, the Abel-Jacobi image is dense in $J^2(X_t)$. But it is well known that the lines in X_t for general $t \in \mathbb{P}^1$, deform in the universal family of quintic threefolds in \mathbb{P}^4 . The corollary follows from this. \square

Remark 4.3. In light of the conjectures in [G-H], Corollary 4.2 most likely does not generalize to higher degree general hypersurface threefolds. However there is a different kind of generalization that probably holds. Namely, let S be the universal family of smooth threefolds $\{V_t\}_{t \in S}$ of degree d say in \mathbb{P}^4 . Put

$$\mathbf{J}_S := \coprod_{t \in S} J^2(V_t),$$

and

$$\mathbf{J}_{S,\mathrm{inv}}^2 := \mathrm{Image}\left(\coprod_{t \in S} \mathrm{CH}_{\mathrm{hom}}^2(V_t) \xrightarrow{\text{Abel-Jacobi}} \mathbf{J}_S\right).$$

Then in the strong topology, we anticipate that $\mathbf{J}_{S,\mathrm{inv}}^2 \subset \mathbf{J}_S$ is a dense subset.

In this direction, we have the following general result.

Corollary 4.4. *Let $\coprod_{\lambda \in S_0} W_\lambda \rightarrow S_0$ be a smooth proper family of $2m$ -dimensional projective varieties in some \mathbb{P}^N with the following property:*

There exists a dense subset $\Sigma \subset S_0$ such that $\lambda \in \Sigma \Rightarrow \mathrm{Prim}^{m,m}(W_\lambda, \mathbb{Q}) \neq 0$, where Prim is primitive cohomology with respect to the embedding $W_\lambda \subset \mathbb{P}^N$. Further, let

$$T := \{t := (c, \lambda) \in \mathbb{P}^{N,*} \times S_0 \mid V_t := \mathbb{P}_c^{N-1} \cap W_\lambda \text{ smooth, \& dim } V_t = 2m-1\},$$

with corresponding $\mathbf{J}_{T,\mathrm{inv}}^m \subset \mathbf{J}_T^m$. Then in the strong topology $\mathbf{J}_{T,\mathrm{inv}}^m$ is dense in \mathbf{J}_T^m .

Proof. This easily follows from the techniques of this section and is left to the reader. \square

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