Rational Self Maps of Calabi-Yau Manifolds

Xi Chen

Abstract. We prove that a very general Calabi-Yau (CY) complete intersection in $\mathbb{P}^n$ does not admit a nontrivial dominant rational self map, assuming that the same holds for very general $K3$ surfaces of genus 3, 4 and 5. One of the crucial steps of our proof makes use of Mumford’s result on the Chow ring of zero-dimensional cycles on a surface with nontrivial holomorphic 2-forms.

1. Introduction

1.1. Statements of results. The main purposes of this paper is to prove the following:

**Theorem 1.1.** There is no dominant rational self map $\phi : X \to X$ of degree $\deg \phi > 1$ for a very general complete intersection $X$ in $\mathbb{P}^n$ of dimension $\dim X \geq 2$ and of type $(d_1, d_2, \ldots, d_r)$ satisfying $d_1 + d_2 + \ldots + d_r \geq n + 1$.

The proof is based on a degeneration argument by “splitting” $X$, introduced by P. Griffiths and J. Harris in [G-H], along with induction on $\dim X$. Eventually, it is reduced to the case $\dim X = 2$, i.e., the case of $K3$ surfaces. We assume the following:

**Theorem 1.2.** There is no dominant rational self map $\phi : X \to X$ of degree $\deg \phi > 1$ for a very general projective $K3$ surface $X$ of genus $3 \leq g \leq 5$, i.e., a very general complete intersection of type $(4)$ in $\mathbb{P}^3$, $(2,3)$ in $\mathbb{P}^4$ or $(2,2,2)$ in $\mathbb{P}^5$.

This was proved in [Ch] for $K3$ surfaces of all genus $g \geq 2$, although we only need $g = 3, 4, 5$ for Theorem 1.1. Indeed, most techniques needed for higher dimensions have already developed in [Ch]. However, this paper is self-contained with no other statement from [Ch] assumed except Theorem 1.2.

If we use the notations $\text{Rat}(X) \supset \text{Bir}(X) \supset \text{Aut}(X)$ for the monoid of dominant rational self maps $\phi : X \to X$ of degree $\phi > 1$, the group of birational self maps $\varphi : X \to X$ and the automorphism group of $X$, respectively, the above theorem is equivalent to saying that

\begin{equation}
\text{Rat}(X) = \text{Bir}(X)
\end{equation}

for a very general complete intersection $X \subset \mathbb{P}^n$ of CY or general type.
In addition, it is well known that a birational map $X \to X$ induces an isomorphism in codimension one, i.e., $\phi : X / Z_1 \cong X / Z_2$ is an isomorphism for some $Z_1$ and $Z_2$ of codimension $\geq 2$ in $X$, since $X$ is smooth and $K_X$ is numerically effective (nef) (see e.g. [Co], 2, (2.5)). It follows that $\phi$ induces an automorphism of the Picard group together with an isomorphism $\phi^* : \Pic(X) \cong \Pic(X)$ of the corresponding linear systems for each $L \in \Pic(X)$. Combining this with the fact $\Pic(X) = \mathbb{Z}$ by Lefschetz, we conclude that $\phi^* = \text{id}$ on $\Pic(X)$, i.e., $\phi^* L = L$ for all $L \in \Pic(X)$ and $\phi^* : \Pic(X) \cong \Pic(X)$ is an automorphism of $\Pic(X)$. Consequently, when we embed $X$ into $\mathbb{P}^N$ with a very ample $L$, $\phi$ is induced by an automorphism of $\mathbb{P}^N$ and hence $\phi \in \text{Aut}(X)$. This proves $\text{Bir}(X) = \text{Aut}(X)$ for a smooth projective variety with $K_X$ nef and $\Pic(X) = \mathbb{Z}$. Hence

$$1.2 \quad \text{Rat}(X) = \text{Bir}(X) = \text{Aut}(X)$$

for a very general complete intersection $X \subset \mathbb{P}^n$ of CY or general type.

Furthermore, since it is classically known that $\text{Aut}(X)$ is trivial for “almost” all complete intersections [M-M], we may put Theorem 1.1 in the following more explicit form:

**Corollary 1.3.** For a very general complete intersection $X \subset \mathbb{P}^n$ of CY or general type and $\dim X \geq 2$,

$$1.3 \quad \text{Rat}(X) = \text{Bir}(X) = \text{Aut}(X) = \{1\}.$$

**1.2. Complete intersections of general type.** Although our theorem is stated for both complete intersections of CY and general type, the only nontrivial part is the statement on CY complete intersections. The theorem is well known to be true for complete intersections of general type. We give a quick proof of this fact.

**Proposition 1.4.** Let $X$ be a smooth projective variety of general type with Hodge group

$$1.4 \quad H^{1,1}(X, \mathbb{Q}) = H^2(X, \mathbb{Q}) \cap H^{1,1}(X) \cong \mathbb{Q}.$$

Then $\deg \phi = 1$ for every dominant rational map $\phi : X \to X$.

**Proof.** The indeterminacy of $\phi$ can be resolved by a sequence of blowups along smooth centers by Hironaka’s resolution of singularities [H] (see also e.g. [K]). Let $f : Y \to X$ be the resulting birational regular map with the commutative diagram

$$1.5 \quad \begin{array}{ccc}
Y & \xrightarrow{\varphi} & X \\
\downarrow f & & \downarrow \phi \\
X & & 
\end{array}$$

where $Y$ is smooth and projective.

From this diagram, we derive the identity

$$1.6 \quad K_Y = f^* K_X + \sum a(E_k, X) E_k = \varphi^* K_X + \sum \mu_l F_l$$

where $K_X$ and $K_Y$ are the canonical divisors of $X$ and $Y$, respectively, $a(E_k, X)$ is the discrepancy of the exceptional divisor $E_k$ with respect to $X$ and $F_l$ are the ramification divisors of $\varphi$. 
By (1.6), we have
\[ K_X = f_* \varphi^* K_X + \sum \mu_i f_* F_i \tag{1.7} \]
and hence
\[ f_* \varphi^* K_X = \lambda K_X \tag{1.8} \]
in \( H^2(X, \mathbb{Q}) \) for some \( 0 < \lambda \leq 1 \), by (1.4) and the fact \( \mu_i > 0 \). Thus
\[ K_X \cdot \varphi_* f_* \xi = f_* \varphi^* K_X \cdot \xi = \lambda K_X \cdot \xi \tag{1.9} \]
for all \( \xi \in H_2(X, \mathbb{Z}) \). Clearly, this forces \( \lambda = \deg \varphi = 1 \) since \( K_X \neq 0 \).

For a smooth complete intersection \( X \) of general type in \( \mathbb{P}^n \), (1.4) holds trivially for \( \dim X = 1 \), for \( \dim X \geq 3 \) by Lefschetz and for \( \dim X = 2 \) and \( X \) very general by Noether-Lefschetz. Hence Theorem 1.1 actually holds for every smooth complete intersection \( X \subset \mathbb{P}^n \) of general type and \( \dim X \geq 3 \).

1.3. Background. For the background on rational self maps of K3 surfaces and CY manifolds, please see the well-written paper [D].

Clearly, every variety \( X \) birational to a projective family of abelian varieties or finite quotients of abelian varieties over some base \( B \) admits nontrivial rational self maps given by the rational self maps of the generic fiber of \( X/B \). Indeed, this observation is one of the main motivations of studying the rational self maps in the first place. As a consequence of Theorem 1.1, we see that a very general CY complete intersection \( X \subset \mathbb{P}^n \) is not birational to a fibration of abelian varieties, which is a known fact for K3 surfaces.

In higher dimensions, C. Voisin proved that a very general CY hypersurface \( X \subset \mathbb{P}^n \) cannot be covered by abelian varieties of dimension \( r \geq 2 \) [V3], which is a stronger statement than \( X \) is not birational to a fibration of abelian varieties of dimension \( r \geq 2 \). However, the case \( r = 1 \) is not known, to the best of our knowledge. Indeed, we expect the following to hold:

**Conjecture 1.5.** For a very general complete intersection \( X \subset \mathbb{P}^n \) of CY or general type and \( \dim X \geq 3 \),
\[ \text{Rat}(Y) = \text{Bir}(Y) \tag{1.10} \]
for every projective variety \( Y \) dominating \( X \) via a generically finite rational map \( Y \rightarrow X \).

Obviously, this conjecture implies that a very general CY complete intersection \( X \subset \mathbb{P}^n \) of dimension \( \geq 3 \) cannot be covered by elliptic curves, which is the weak Clemens’ conjecture when \( X \) is a very general quintic 3-fold.

A bolder conjecture is that everything here including Theorem 1.1 and Conjecture 1.5 holds for a very general projective CY manifold. Certainly, the techniques developed here can be easily adapted to deal with other types of CY manifolds as long as these manifolds admit suitable degenerations. On the other hand, Voisin gives examples of CY varieties with Picard number one having dominant rational self maps of degree \( > 1 \) [V2, Sec. 4.2] (see also [D-M, Theorem 3.4]).

Correspondingly, there is a similar story for generalizations of Clemens’ conjecture [V1, Remark 3.24].
1.4. Conventions and Acknowledgments. We work exclusively over $\mathbb{C}$ and with analytic topology wherever possible.

By a component of a variety, we mean an irreducible component unless we say a connected component of a variety, which is a connected component of the variety in the topological sense.

I am grateful to Prof. Keiji Oguiso for point it out to me the fact that $\text{Bir}(X) = \text{Aut}(X)$ for a Calabi-Yau manifold $X$ with Picard rank one. I would also like to thank the referee for many constructive suggestions.

2. Proof of Theorem 1.1

2.1. Degeneration. We start our proof by degenerating a complete intersection of type $(d_1, d_2, \ldots, d_r)$ to a union of two complete intersections of type $(d'_1, d_2, \ldots, d_r)$ and $(d''_1, d_2, \ldots, d_r)$, respectively. More precisely, let $W \subset \Delta \times \mathbb{P}^n$ be a family of complete intersections of type $(d_1, d_2, \ldots, d_r)$ over the disk $\Delta$ with the properties that

- the central fiber $W_0 = S_1 \cup S_2$, where $S_1$ and $S_2$ are smooth complete intersections of type $(d'_1, d_2, \ldots, d_{r-1}, d_r)$ and $(d''_1, d_2, \ldots, d_{r-1}, d_r)$, respectively, for some $d'_1, d''_1 \in \mathbb{Z}^+$ satisfying $d'_1 + d''_1 = d_1$;
- $S_1$ and $S_2$ meet transversely along $D = S_1 \cap S_2$, where $D$ is a very general complete intersection of type $(d'_1, d''_1, d_2, \ldots, d_{r-1}, d_r)$ in $\mathbb{P}^n$ and hence

\begin{equation}
\text{Rat}(D) = \text{Bir}(D) = \text{Aut}(D) = \{1\}
\end{equation}

by induction;
- $W$ is smooth outside of a smooth complete intersection $\Lambda \subset D$ of type $(d'_1, d''_1, d_2, \ldots, d_{r-1}, d_r)$ in $\mathbb{P}^n$ and is locally given by $xy = tz$ at every point $p \in \Lambda$.

We can resolve the singularities of $W$ by blowing up $W$ along $S_1$. Let $X \to W$ be the blowup. It is not hard to see that the central fiber of $X/\Delta$ is $X_0 = R_1 \cup R_2$, where $R_1$ is the blowup of $S_1$ at $\Lambda$ and $R_2 \cong S_2$.

2.2. Resolution of indeterminacy. Suppose that there is a dominant rational map $\phi_t : X_t \dashrightarrow X_t$ of $\deg \phi_t > 0$ for every $t \neq 0$. We can extend it to a dominant rational map $\phi : X \dashrightarrow X$, after a base change, with the commutative diagram

\begin{equation}
\begin{array}{ccc}
X & \xrightarrow{\phi} & X \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{\phi} & \Delta
\end{array}
\end{equation}

Note that after a base change, $X$ is locally given by

\begin{equation}
xy = t^m
\end{equation}

for some positive integer $m$ at every point $p \in D$. So $X$ is $\mathbb{Q}$-factorial and has canonical singularities along $D$.

As in the proof of Proposition 1.4 we can resolve the indeterminacy of $\phi$ and arrive at the diagram (1.5), while preserving the base $\Delta$. In addition, we can make $Y_0$ into a divisor with simple normal crossings after a further base change by the stable reduction theorem in [KKMS].
Let $\omega_{X/\Delta}$ and $\omega_{Y/\Delta}$ be the relative dualizing sheaves of $X$ and $Y$ over $\Delta$, respectively. We have the family version of (1.6):

\begin{equation}
\omega_{Y/\Delta} = f^*\omega_{X/\Delta} + \sum a(E_k, X)E_k = \varphi^*\omega_{X/\Delta} + \sum \mu_i F_i
\end{equation}

which plays a central role in our argument.

Since we have proved the theorem for $X_t$ of general type, we assume that $X_t \subset \mathbb{P}^n$ is a CY complete intersection. In this case, we have $\sum a(E_k, X)E_k = \sum \mu_i F_i$ and (2.4) becomes

\begin{equation}
\omega_{Y/\Delta} = \sum \mu(E)E = \sum \mu(E)E = \varphi^*\omega_{X/\Delta} + \sum \mu(E)E.
\end{equation}

where $\mu$ is the function defined by $\mu(E) = a(E, X)$ for all irreducible divisors $E \subset Y$ satisfying $f, E = 0$. For convenience, we let $\mu(\tilde{R}_i) = 0$ for $i = 1, 2$, where $\tilde{R}_i \subset Y$ are the proper transforms of $R_i$ under $f$.

Since $X$ has at worse canonical singularities, we see that $\mu(E) \geq 0$ for all $E$. And we claim the following:

**Proposition 2.1.** For a component $E \subset Y_0$,

\begin{equation}
\mu(E) > 0 \Rightarrow \varphi_* E = 0.
\end{equation}

To see this, we apply the following simple observation.

**Lemma 2.2.** Let $X/\Delta$ and $Y/\Delta$ be two flat families of complex analytic varieties of the same dimension over the disk $\Delta$. Suppose that $X$ has reduced central fiber $X_0$ and $Y$ is smooth. Let $\varphi : Y \to X$ be a proper surjective holomorphic map preserving the base. Let $S \subset Y_0$ be a reduced irreducible component of $Y_0$ with $\varphi_* S \neq 0$. Suppose that $\varphi$ is ramified along $S$ with ramification index $\nu > 1$. Then $S$ has multiplicity $\nu$ in $Y_0$. In particular, $Y_0$ is nonreduced along $S$.

**Proof.** The problem is entirely local. Let $R = \varphi(S)$, $q$ be a general point on $S$ and $p = \varphi(q)$. Let $U$ be an analytic open neighborhood of $p$ in $X$ and let $V$ be the connected component of $\varphi^{-1}(U)$ that contains the point $q$. We may replace $X$ and $Y$ by $U$ and $V$, respectively. Then we reduce it to the case that $R$ and $S$ are the only components of $X_0$ and $Y_0$, respectively, $R$ and $S$ are smooth and $\varphi : S \to R$ is an isomorphism, in which case the lemma follows easily. \qed

**Proof of Proposition 2.1.** If $\varphi_* E \neq 0$, then $\varphi$ is ramified along $E$ with ramification index $\mu(E) + 1$ by (2.4) and Riemann-Hurwitz. This is impossible unless $\mu(E) = 0$ by the above lemma and the fact that $Y_0$ is of simple normal crossing. Consequently, (2.6) follows. \qed

We let $S \subset Y_0$ be the union of components $E$ with $\mu(E) = 0$, i.e.,

\begin{equation}
S = \sum_{\mu(E) = 0} E.
\end{equation}

Then it follows from (2.6) that

\begin{equation}
\varphi_* S = (\deg \phi)(R_1 + R_2).
\end{equation}

Since $X$ is smooth outside of $D$, $\mu(E) > 0$ if $f(E) \not\subset D$ and $f_* E = 0$. Consequently, we have $f(E) \subset D$ for every component $E \subset S$ with $f_* E = 0$. Note that $\tilde{R}_i \subset S$. 

Actually, we can arrive at a precise picture of $S$ as follows.

### 2.3. Structure of $S$.

We may resolve the singularities of $X$ by repeatedly blowing up $X$ along $R_1$. By that we mean we first blow up $X$ along $R_1$, then we blow up the proper transform of $R_1$ and so on. Let $\eta : X' \to X$ be the resulting resolution. We see that

$$X'_0 = P_0 \cup P_1 \cup \ldots \cup P_{m-1} \cup P_m$$

where $P_0$ and $P_m$ are the proper transforms of $R_1$ and $R_2$, respectively, $P_i$ are $\mathbb{P}^1$ bundles over $D$ for $0 < i < m$ and $P_i \cap P_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Note that the relative dualizing sheaf of $X'/\Delta$ satisfies

$$\omega_{X'/\Delta} = \eta^* \omega_{X/\Delta}$$

and hence remains trivial.

We have the commutative diagram

$$Y \xrightarrow{\varphi} X \xrightarrow{\eta} X'$$

$$\nu \downarrow \downarrow \phi$$

where $\nu = \eta^{-1} \circ f$. By (2.10), $a(P_i, X) = 0$ for all $i$. Note that the discrepancy $a(P_i, X)$ does not depend on the birational model of $X$. So we necessarily have the proper transform $\nu^{-1}(P_i) \neq 0$. Otherwise, $P_i$ would be the proper transform of an exceptional divisor of some birational regular map $Y' \to Y$ and hence $a(P_i, Y) = a(P_i, Y) > 0$ since $Y$ is smooth. Contradiction. Consequently, there exist $Q_i \subset Y_0$ which are the proper transforms of $P_i$ under $\nu$ for $i = 0, 1, \ldots, m$.

On the other hand, for every component $Q \subset Y_0$ with $Q \notin \{Q_0, Q_1, \ldots, Q_m\}$, we have $\nu_* Q = 0$ and hence $a(Q, X) > 0$ by the same argument as above. Therefore, $Q_i$ are the only components of $Y_0$ with $\mu(Q_i) = 0$. Consequently,

$$S = Q_0 + Q_1 + \ldots + Q_{m-1} + Q_m,$$

$$f(Q_i) = D \text{ for } 0 < i < m,$$

and

$$\varphi_* S = \sum_{i=0}^m \varphi_* Q_i = (\deg \phi)(R_1 + R_2).$$

Obviously, $Q_i$ is birational to $D \times \mathbb{P}^1$ for each $0 < i < m$.

Note that $Q_0 = \bar{R}_1$ and $Q_m = \bar{R}_2$.

Let $T$ be a component of $Y_0$. Then by (2.5) and adjunction, we have

$$\omega_T = (\omega_{Y/\Delta} + T) \bigg|_T = \sum_{E \in Y_0} \mu_E E \bigg|_T - \sum_{E \subseteq Y_0} E \bigg|_T$$

where we write $\mu_E = \mu(E)$; here we use the fact that $T = -(Y_0 - T)$ as $\text{Pic}(\Delta) = 0$. Hence

$$\sum_{E \subseteq Y_0} \mu_E E \bigg|_T = \omega_T + \sum_{E \subseteq Y_0} (1 + \mu_T - \mu_E) E \bigg|_T$$
Suppose that $T = Q \subset S$. Then (2.16) becomes
\begin{equation}
\sum_{E \not\subset Q} (1 - \mu_E)E \bigg|_Q = -\omega_Q + \sum_{E \not\subset Y_0} \mu_E E \bigg|_Q
\end{equation}
Suppose that $Q \neq Q_0, Q_m$. Let $F_p \cong \mathbb{P}^1$ be the fiber of $f : Q \to D$ over a general point $p \in D$. Clearly, we have
\begin{equation}
F_p \cdot \omega_Q = -2
\end{equation}
and hence
\begin{equation}
\sum_{E \not\subset Q} (1 - \mu_E)E \cdot F_p \geq 2.
\end{equation}
Therefore, each $Q_j (0 < j < m)$ meets at least two other $Q_i (0 \leq i \leq m)$ along rational sections of $Q_j/D$; and since $Q_j$ is the proper transform of $P_j$, it cannot meet more than two among $Q_i$. So we see that $Q_i$ form a "chain" in the same way as $P_i$ do. More precisely, we have
- $Q_i$ and $Q_{i+1}$ meet transversely along a component $D_i$ of $Q_i \cap Q_{i+1}$ satisfying $f(D_i) = D$ for $0 \leq i < m$;
- $D_i$, birational to $D$, are rational sections of $f : Q_i \to D$ for $1 \leq i \leq m - 1$ and $f : Q_{i+1} \to D$ for $0 \leq i \leq m - 2$.
- $f(Q_i \cap Q_j) \neq D$ for $|i - j| > 1$.
Next, we claim that
\begin{equation}
\text{Proposition 2.3. For each } 0 \leq i \leq m, \text{ we have}
\end{equation}
\begin{equation}
\text{either } \varphi_* Q_i \neq 0 \text{ or } \varphi(Q_i) = D.
\end{equation}
Namely, every $Q_i$ either dominates one of $R_1$ and $R_2$ or is contracted onto $D$ by $\varphi$. Since $\tilde{R}_i$, being Fano, cannot be mapped onto $D$, which is a CY manifold, this implies that
\begin{equation}
\varphi_* Q_0 \neq 0 \text{ and } \varphi_* Q_m \neq 0.
\end{equation}
So $\varphi$ does not contract either $Q_0 = \tilde{R}_1$ or $Q_m = \tilde{R}_2$.
Note that if $X$ were smooth, we would already have that $\varphi_* S \neq 0$ for all $S$ with $\mu(S) = 0$ by (2.5) and Riemann-Hurwitz. However, things are a little more subtle here since $X$ is singular.

PROOF OF PROPOSITION 2.3 A natural thing to do is to resolve the indeterminacy of the rational map $\phi' = \eta^{-1} \circ \phi \circ \eta : X' \dashrightarrow X'$ with the diagram
\begin{equation}
\begin{array}{c}
Y' \\
\downarrow \varphi' \\
X' \\
\downarrow \eta \\
X \\
\end{array}
\begin{array}{c}
\downarrow \phi' \\
\downarrow \phi \\
\end{array}
\begin{array}{c}
\nu' \\
\end{array}
\begin{array}{c}
\nu \\
\end{array}
\end{equation}
where we can make $Y'_0$ into a divisor of simple normal crossing support $[H]$. Let $Q_i' \subset Y'$ be the proper transforms of $P_i$ under $\nu'$. Obviously, $Q'_i$ are the proper transforms of $Q_i$ under the birational map $\nu^{-1} \circ \nu' : Y' \dashrightarrow Y$. To show that (2.20)
holds for $Q_i$, it suffices to show that the same thing holds for $Q'_i$ when we map $Y'$
to $X$ via $\eta \circ \varphi'$.

We have

\[
\omega_{Y'/\Delta} = (\nu')^*\omega_{X/\Delta} + \sum a(E, X)E
\]

where $E$ runs through all exceptional divisors of $\eta \circ \nu'$. By (2.10), we see that
$a(Q'_i, X) = 0$ for all $0 \leq i \leq m$. Then we have $\varphi'_i Q'_i \neq 0$ by Riemann-Hurwitz and
the fact that $X'$ is smooth. So each $Q'_i$ dominates some $P_j$ via $\varphi'$. If $Q'_i$ dominates
$P_0$ or $P_m$, then $Q'_i$ dominates $R_1$ or $R_2$ via $\eta \circ \varphi'$; if $Q'_i$ dominates $P_j$ for some
$0 < j < m$, then $\eta(\varphi'(Q'_i)) = D$. This proves (2.20).

**Corollary 2.4.** Let $0 \leq i < j \leq m$ be two integers with the properties that
$\varphi, Q_i \neq 0$, $\varphi, Q_j \neq 0$ and $\varphi, Q_k = 0$ for all $i < k < j$. Then
\[
\varphi, D_k = D \text{ and } \varphi, D_k \circ f^{-1}_{D_k} = 1
\]
for all $i \leq k < j$, where $\varphi, D_k : D_k \to D$ and $f_{D_k} : D_k \to D$ are the restrictions of
$\varphi$ and $f$ to $D_k$, respectively. In particular,
\[
\deg \varphi, D_k = 1
\]
for all $0 \leq k < m$, i.e., $\varphi$ maps each $D_k$ birationally onto $D$.

**Remark 2.5.** This is the only place where the induction hypothesis is needed: we need it to show that the dominant rational self map $\varphi_{D_k} \circ f^{-1}_{D_k} : D \dashrightarrow D$ is
an identity, while $D \subset \mathbb{P}^n$ is a very general complete intersection CY manifold of
lower dimension. Eventually, we will reduce Theorem 1.1 to the case of complete intersection $K3$ surfaces where Theorem 1.2 is required.

**Proof of Corollary 2.3.** If $\varphi, Q_k = 0$, $\varphi$ maps $Q_k$ onto $D$ by Proposition
since $D$ is a CY manifold, $\varphi$ must contract the fibers of $f : Q_k \to D$. Therefore,$\varphi(D_{k-1}) = \varphi(D_k) = D$. Hence $\varphi, D_{k-1} \circ f^{-1}_{D_{k-1}}$ and $\varphi, D_k \circ f^{-1}_{D_k}$ are dominant
rational self maps of $D$; by induction hypothesis (2.1), they must be identity maps. Therefore, (2.24) follows easily when $j - i > 1$.

When $j - i = 1$, we will reduce it to the case $j - i > 1$ by applying a further base change. That is, for some $a > 1$, there is $Y'$ with the commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{\delta} & Y'^{[a]} \\
\downarrow f' & & \downarrow f' \\
X'^{[a]} & \xrightarrow{f} & Y^{[a]}
\end{array}
\]

and all the required properties, where $X'^{[a]} = X \otimes \mathbb{C}[[\sqrt{\overline{t}}]]$ and $Y'^{[a]} = Y \otimes \mathbb{C}[[\sqrt{\overline{t}}]]$.
Correspondingly, $S' = Q'_0 + Q'_1 + ... + Q'_{ma}$ with $Q'_{ka}$ the proper transform of $Q_k$.
Obviously, $\varphi, Q'_{ia} \neq 0$, $\varphi, Q'_{ia} \neq 0$ and $\varphi, Q'_{ia} = 0$ for all $ia < k < ja$ and $\varphi' = \varphi \circ \delta$. It is also clear that (2.24) holds for $(Y', f', \varphi')$ if it holds for $(Y', f', \varphi')$. Hence it again follows easily from Proposition 2.3 as $ja - ia > 1$.

We might want to “get rid of” $Q_i$’s that are contracted by $\varphi$. Let $X'$ be the
variety obtained from $X$ by contracting all components $P_i$ with $\varphi, Q_i = 0$. After a possible further base change and stable reduction, we may assume that $f : Y \to X$
factors through $X'$; namely, we resolve the indeterminacy of $\theta^{-1} \circ f : Y \dashrightarrow X'$ and then apply stable reduction so that $Y_0$ remains a divisor with simple normal
crossings, where $\theta$ is the map $X \to X$ factored through by $\eta : X' \to X$. 
Then we have the commutative diagram

\[
\begin{array}{cccc}
X' & \rightarrow & X & \rightarrow & X \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi \\
Y & \rightarrow & W & \rightarrow & W
\end{array}
\]

The choice of \( \mathcal{X} \) guarantees that

\[
(\phi \circ \varepsilon^{-1})_{\mathcal{X}_0} = (\deg \phi)X_0.
\]

We continue to denote the components of \( \mathcal{X}_0 \) by \( P_i \). So \( \mathcal{X}_0 \) is the unions of \( P_i \) for \( i \) satisfying \( \varphi_* Q_i \neq 0 \).

Using the diagram (2.27), we can prove the following:

**Proposition 2.6.** Let \( Q = Q_i \subset S \) be a component satisfying \( \varphi_* Q \neq 0 \) and let \( \Gamma \subset Q \) be an irreducible subvariety of codimension one in \( Q \) such that \( \varphi(\Gamma) \subset D \).

Then either \( \Gamma \) is one of \( D_{i-1} \) and \( D_i \) or \( \varphi(\Gamma) = 0 \).

**Proof.** Otherwise, \( \Gamma \neq D_{i-1} \), \( \Gamma \neq D_i \) and \( \varphi(\Gamma) = D \). So \( \Gamma \) has nonnegative Kodaira dimension and hence \( \varepsilon, \Gamma \neq 0 \). Let \( G = \varepsilon(\Gamma) \). Since \( \varphi_* Q \neq 0 \), \( \varepsilon_* Q \neq 0 \) by our construction of \( \mathcal{X} \). So \( P = \varepsilon(Q) \) is a component of \( \mathcal{X}_0 \). And since \( \varepsilon : Q \to P \) is birational, \( G \neq \varepsilon(D_i) \). That is, \( G \nsubseteq P' \) for all components \( P' \neq P \) of \( \mathcal{X}_0 \).

Therefore, \( \varepsilon^{-1}(G) \) does not contain any component of \( S \).

Let \( \Sigma \) be the union of components of \( Y_0 \) that dominate \( G \) via \( \varepsilon \) and let \( q \) be a general point on \( \Gamma \), \( p = \varepsilon(q) \) and \( J = \varepsilon^{-1}(p) \). By Zariski’s main theorem, \( J \) is connected. If \( \dim J = 0 \), then \( J = \{q\} \) and \( \Sigma = \emptyset \).

Suppose that \( \dim J = 1 \). Since \( J \) is connected, \( \Sigma \) is connected. Let \( J_1 \subset J \) be the component of \( J \) containing \( q \) and let \( \Sigma_1 \subset \Sigma \) be the component of \( Y_0 \) containing \( J_1 \). Obviously, \( \Gamma \subset \Sigma_1 \) and hence \( D \subset \varphi(\Sigma_1) \). And since \( \Sigma_1 \nsubseteq S \), \( \varphi_* \Sigma_1 = 0 \). Therefore, \( \varphi(\Sigma_1) = D \). And since \( J_1 \cong \mathbb{P}^1 \subset \Sigma_1 \), \( \varphi_* J_1 = 0 \) and \( \varphi \) contracts \( \Sigma_1 \) onto \( D \) along the fibers of \( \varepsilon : \Sigma_1 \to G \). Let \( J_2 \neq J_1 \subset J \) be a component of \( J \) with \( J_1 \cap J_2 \neq \emptyset \) and let \( \Sigma_2 \subset \Sigma \) be the component of \( Y_0 \) containing \( J_2 \). Then \( \Sigma_2 \) meets \( \Sigma_1 \) along a rational multi-section of \( \Sigma_1/G \). Therefore, \( \varphi(\Sigma_2) = D \) and \( \varphi_* J_2 = 0 \) by the same argument as before. We can argue this way inductively that \( \varphi_* J = 0 \) and \( \varphi(\Sigma^o) = D \) for every component \( \Sigma^o \subset \Sigma \).

Let \( r = \varphi(q) \) and \( K \) be the connected component of \( \varphi^{-1}(r) \) containing the point \( q \). Obviously, \( J \subset K \). We claim that \( J = K \). Otherwise, there is a component \( K^o \subset K \) such that \( K^o \nsubseteq J \) and \( K^o \cap J \neq \emptyset \). Let \( T \) be a component of \( Y_0 \) containing \( K^o \). Obviously, \( T \nsubseteq \Sigma \); otherwise, we necessarily have \( \varepsilon(K^o) = p \) and \( K^o \subset J \). Also we cannot have \( T = Q \); otherwise, \( K^o \subset Q \), \( q \in K^o \) and \( \varphi, K^o = 0 \), which is impossible for a general point \( q \in \Gamma \). We cannot have \( T = Q' \) for some \( Q' \neq Q \subset S \), either, since \( p \in \varepsilon(K^o) \subset \varepsilon(T) \). Therefore, \( T \nsubseteq S \).

If \( J = \{q\} \), then \( q \in K^o \) since \( K^o \cap J \neq \emptyset \); it follows that \( \Gamma \subset T \) and \( \varepsilon(T) = G \), which is impossible since \( \Sigma = \emptyset \).

Otherwise, suppose that \( \dim J = 1 \). Again since \( K^o \cap J \neq \emptyset \), \( T \cap \Sigma \neq \emptyset \). If \( T \) and \( \Sigma \) meet along a rational multi-section of \( \Sigma/G \), \( \varepsilon(T) = G \), which is impossible as we have proved that \( T \nsubseteq \Sigma \). Therefore, \( T \cap \Sigma \) is contained in the fibers of \( \Sigma/G \). And since \( T \cap J \neq \emptyset \), \( T \cap \Sigma \) contains a component of \( J \), which is impossible for a general point \( p \in G \). Therefore, \( J = K \).

Let \( U \subset X \) be an analytic open neighborhood of \( r \) in \( X \) and \( V \subset Y \) be the connected component of \( \varphi^{-1}(U) \) containing \( J \). Since \( q \) is a general point of \( \Gamma \),
\[ \varepsilon(J) = p \notin P_j \text{ for all } P_j \subset X_0 \text{ and } j \neq i. \] Consequently, \( V \cap S = V \cap Q \). Therefore, \( \varphi_M = 0 \) for all components \( M \) of \( V_0 \) with \( M \neq V \cap Q \). So \( V \) cannot dominate \( U \).

\[ \square \]

### 2.4. Invariants \( \alpha_i \) and \( \beta_i \)

Let \( Q_i \) be a component of \( S \). Suppose that \( Q_i \) dominates \( R_j \) via \( \varphi \) for some \( 1 \leq j \leq 2 \). Let \( \varphi_{Q_i} : Q_i \to R_j \) be the restriction of \( \varphi \) to \( Q_i \). Proposition 2.6 tells us that \( D_{i-1} \) and \( D_i \) are the only components of \( \varphi_{Q_i}^{-1}(D) \) that dominate \( D \) via \( \varphi \).

Let \( \alpha_i \) and \( \beta_{i-1} \) be the ramification indices of \( \varphi_{Q_i} \) along \( D_{i-1} \) and \( D_i \), respectively, where we set \( D_{-1} = D_m = \emptyset \), \( \alpha_m = \beta_{-1} = 0 \) and \( \alpha_i = \beta_{i-1} = 0 \) if \( \varphi_{Q_i} = 0 \). Since \( \varphi_{Q_i}, \varphi_{Q_i}^2, D = (\deg \varphi_{Q_i})D_i \),

\begin{equation}
\deg \varphi_{Q_i} = \alpha_i \deg \varphi_{D_i} + \beta_{i-1} \deg \varphi_{D_{i-1}} = \alpha_i + \beta_{i-1}
\end{equation}

where \( \deg \varphi_{D_{i-1}} = \deg \varphi_{D_i} = 1 \) by (2.25).

Actually \( \alpha_i \) and \( \beta_{i-1} \) are very explicitly determined as follows.

**Proposition 2.7.** Let \( 0 \leq i < j \leq m \) be two integers with the properties that \( \varphi_{Q_i} \neq 0, \varphi_{Q_j} \neq 0 \) and \( \varphi_{Q_k} = 0 \) for all \( k < i < j \). Then

\begin{equation}
\alpha_i = \beta_{j-1} = \frac{m}{j-i}
\end{equation}

and \( \varphi(\xi_i) \neq \varphi(\xi_j) \).

**Proof.** Let \( q \) be a general point on \( D_i \) and \( U \subset X \) be an analytic open neighborhood of \( \varphi(q) \) in \( X \). Let \( V \subset Y \) be the connected component of \( \varphi^{-1}(U) \) containing \( q \). Let \( T \subset V_0 \) be a component of \( V \cap Y_0 \) satisfying \( T \not\subset S \). Since \( q \) is a general point on \( D_i \), it is easy to see that \( \varepsilon(T) = P_i \cap P_j \). Indeed, \( T \) is a \( \mathbb{P}^1 \)-bundle over \( \varepsilon(T) \) and \( \varphi \) contracts the fibers of \( T/\varepsilon(T) \) and maps \( T \) onto \( D_i \cap U \).

Therefore, \( \varepsilon : V \to V' \equiv \varepsilon(V) \) is proper and \( V' \) is open in \( X \). In addition, since \( T \) is contracted by \( \varphi \) along the fibers of \( T/\varepsilon(T) \), the rational map \( \varphi \circ \varepsilon^{-1} : V' \to U \) is actually regular.

Locally, at \( \varepsilon(q) \in P_i \cap P_j \), \( V' \subset X \) is given by \( xy = t^{j-i} \) in the polydisk \( \Delta_{x_1y_2...t}^N \) with \( P_i = \{ x = 0 \} \) and \( P_j = \{ y = 0 \} \). Similarly, at \( \varphi(q) \in D_i \subset X \) is given by \( xy = t^m \) in \( \Delta_{x_1y_2...t}^N \) with \( R_1 = \{ x = 0 \} \) and \( R_2 = \{ y = 0 \} \). The map \( \varphi \circ \varepsilon^{-1} \) is regular and finite and sends \( V' = \{ xy = t^{j-i} \} \) onto \( U = \{ xy = t^m \} \) while preserving the base \( \Delta = \{|t| < 1 \} \). It has to be the map sending \((x,y,z,...,t)\) to \((x^a,y^a,z,...,t)\) or \((y^a,x^a,z,...,t)\) with \( a = m/(j-i) \). It follows that \( \alpha_i = \beta_{j-1} = a \) and \( \varphi(\xi_i) \neq \varphi(\xi_j) \).

**Corollary 2.8.** The following holds:

- \( \alpha_i \neq 1 \) and \( \beta_{i-1} \neq 1 \) for all \( 0 < i < m \).
- If \( \deg \varphi_{Q_0} = 1 \) or \( \deg \varphi_{Q_m} = 1 \), then \( \deg \varphi = 1 \).

**Proof.** The first statement follows directly from Proposition 2.7.

If \( \deg \varphi_{Q_0} = 1 \), then \( \alpha_0 = \deg \varphi_{D_0} = 1 \). By Proposition 2.7, we must have \( \beta_{m-1} = 1 \) and \( \varphi_{Q_k} = 0 \) for all \( 0 < k < m \). Hence \( \deg \varphi_{D_m} = \deg \varphi_{D_0} = 1 \) and \( \deg \varphi_{Q_m} = 1 \) by (2.29). It follows that \( \deg \varphi = 1 \).

Similarly, we can show that \( \deg \varphi = 1 \) if \( \deg \varphi_{Q_1} = 1 \).

**Corollary 2.9.** The following are equivalent:

- \( \alpha_0 = 1 \).
- \( \beta_{m-1} = 1 \).
• \( \alpha_i = 1 \) for some \( 0 \leq i \leq m - 1 \).
• \( \beta_j = 1 \) for some \( 0 \leq j \leq m - 1 \).
• \( \varphi_i Q_i = 0 \) for all \( 1 \leq i \leq m - 1 \).
• \( \deg \phi_{R_1} = \deg \phi, \) where \( \phi_{R_1} \) is the restriction of \( \phi \) to \( R_1 \).
• \( \deg \phi_{R_2} = \deg \phi, \) where \( \phi_{R_2} \) is the restriction of \( \phi \) to \( R_2 \).
• \( \deg \phi = 1 \).

In particular, at least one of \( Q_i \) (\( i \neq 0, m \)) is not contracted by \( \varphi \) if \( \deg \phi > 1 \).

PROOF. This is more or less trivial. \( \square \)

2.5. The case of hypersurfaces. Suppose that \( \deg \phi > 1 \). Then there exists \( 1 \leq i \leq m - 1 \) such that \( Q = Q_i \) dominates \( R_j \) via \( \varphi \) for some \( 1 \leq j \leq 2 \) by Corollary 2.9.

To make our life a little easier, we can set \( j = 2 \) by replacing \( \phi \) with \( \phi^2 \) if necessary and applying the following observation:

\begin{equation}
\phi_*(R_1 + R_2) \neq (\deg \phi)R_j.
\end{equation}

PROOF. It is easy to see by (2.14) that (2.31) holds if \( \varphi(Q_i) = R_j \) for some \( 1 \leq i \leq m - 1 \).

On the other hand, suppose that (2.31) holds. By Proposition 2.7, \( \varphi(Q_i) = R_j \) for some \( 1 \leq i \leq m - 1 \) if \( \phi (R_1) \neq \phi (R_2) \) or \( \phi (R_1) = \phi (R_2) \neq R_j \).

If \( \phi (R_1) = \phi (R_2) = R_j \), then it follows from (2.14) again that \( \varphi(Q_i) = R_j \) for some \( 1 \leq i \leq m - 1 \). \( \square \)

Suppose that (2.31) fails for \( j = 2 \). That is,

\begin{equation}
\phi_*(R_1 + R_2) = (\deg \phi)R_2.
\end{equation}

Let \( \phi^2 = \phi \circ \phi \). It is easy to see that

\begin{equation}
(\phi^2)_*(R_1 + R_2) = (\deg \phi)(\deg \phi)R_2 \neq (\deg \phi^2)R_2
\end{equation}

since \( \deg \phi R_2 \neq \deg \phi \). So if (2.31) fails, it will hold for \( \phi^2 \). In conclusion, by replacing \( \phi \) by \( \phi^2 \) if necessary, we can always find \( 1 \leq i \leq m - 1 \) such that \( Q = Q_i \) dominates \( R_2 \) via \( \varphi \).

So far we have all the geometric facts about \( \varphi Q = \varphi Q : Q \to R_2 \) that we need to complete our proof:

A1. let \( F_p = f_Q^{-1}(p) \) be the fiber of \( f_Q : Q \to D \) over a general point \( p \in D \),

\( p_{i-1} = F_p \cap D_{i-1} \) and \( p_i = F_p \cap D_i \); then

\begin{equation}
\varphi(p_{i-1}) = \varphi(p_i) = p
\end{equation}

due to (2.24);

A2. \( D_{i-1} \) and \( D_i \) are the only components of \( \varphi_Q^{-1}(D) \) that dominates \( D \) via \( \varphi \); consequently, there is a subvariety \( Z \subset \overline{D} \) of codimension \( \geq 1 \), independent of \( p \), such that

\begin{equation}
\varphi_{Q*} F_p : D = \alpha_i \varphi(p_i) + \beta_{i-1} \varphi(p_{i-1}) + \Sigma = (\alpha_i + \beta_{i-1}) p + \Sigma
\end{equation}

with \( \Sigma \) supported on \( Z \), where \( \varphi_{Q*} \) is the push-forward induced by \( \varphi_Q : Q \to R_2 \).
In summary, for \( p \in D \) general, i.e. \( p \) outside of a proper closed subvariety \( Z'' \subset D \), one has

\[
p = \frac{1}{\alpha_i + \beta_{i-1}} (\varphi_{Q_i*}F_p \cdot D) - \frac{1}{\alpha_i + \beta_{i-1}} \Sigma
\]

in \( \text{CH}_0(D, \mathbb{Q}) \) by (2.35), where \( \Sigma \) lies in the image of \( \text{CH}_0(Z, \mathbb{Q}) \to \text{CH}_0(D, \mathbb{Q}) \).

If \( \varphi_{Q_i*}F_p \) lies in the same class of \( \text{CH}_1(R_2, \mathbb{Q}) \) for all \( p \), which happens when

\[
\text{CH}_1(R_2, \mathbb{Q}) = H_2(R_2, \mathbb{Q}) = \mathbb{Q},
\]

then \( \varphi_{Q_i*}F_p \) is a constant class in \( \text{CH}_1(R_2, \mathbb{Q}) \) for all \( p \) and hence

\[
c_0 = \frac{1}{\alpha_i + \beta_{i-1}} (\varphi_{Q_i*}F_p \cdot D)
\]

is a constant class in \( \text{CH}_0(D, \mathbb{Q}) \). We can certainly choose \( Z'' \) such that \( c_0 \) lies in the image of \( \text{CH}_0(Z'', \mathbb{Q}) \to \text{CH}_0(D, \mathbb{Q}) \). It follows that the closed immersion \( i : Z' = Z \cup Z'' \hookrightarrow D \) induces a surjection

\[
i_* : \text{CH}_0(Z', \mathbb{Q}) \to \text{CH}_0(D, \mathbb{Q})
\]

with \( \text{codim}_D Z' \geq 1 \). This cannot happen for a CY manifold \( D \) by Roıţman’s generalization of Mumford’s classical results on Chow rings of zero-dimensional cycles on surfaces [R1] (see also [M1], [R2] and [B-S]):

**Theorem 2.11** (Mumford, Roıţman, Bloch-Srinivas). Let \( X \) be a smooth projective variety of dimension \( n \). If there exists \( i : Y \hookrightarrow X \) such that \( \dim Y < n \) and

\[
i_* : \text{CH}_0(Y, \mathbb{Q}) \to \text{CH}_0(X, \mathbb{Q})
\]

is surjective, then \( h^{n,0}(X) = 0 \).

This gives us a quick proof for hypersurfaces, in particular, quintic 3-folds in \( \mathbb{P}^4 \), since for hypersurfaces of degree \( d_1 \) in \( \mathbb{P}^n \), we can do the splitting \( d'_1 = d_1 - 1 \) and \( d''_1 = 1 \). That is, \( R_2 \cong \mathbb{P}^{n-1} \) is a hyperplane in \( \mathbb{P}^n \) and hence (2.37) holds.

**2.6. Completion of the proof.** For complete intersections, we cannot guarantee (2.37). Indeed, (2.37) is only known for \( d_2, \ldots, d_r \) sufficiently small (we can always take \( d''_1 = 1 \)), in which case \( \text{CH}_1(R) \) is generated by the lines on \( R = R_2 \).

However, we can get around the problem by taking advantage of the fact that \( D \subset R \) is a general member of the complete linear series \( |\mathcal{O}_R(D)| \). Let \( \mathcal{D} \subset |\mathcal{O}_R(D)| \times R \) be the incidence correspondence

\[
\mathcal{D} = \{(D', p) : D' \in |\mathcal{O}_R(D)|, p \in D'\}.
\]

We observe that \( R \) is Fano and hence rationally connected and the projection \( \mathcal{D} \to R \) gives \( \mathcal{D} \) the structure of a \( \mathbb{P}^{N-1} \)-bundle over \( R \), since \( |\mathcal{O}_R(D)| = \mathbb{P}^N \) is base point free. Therefore, \( \mathcal{D} \) is rationally connected and hence

\[
\text{CH}_0(\mathcal{D}) = \mathbb{Z}.
\]

Since we can find \( Q_{D'} \) dominating \( R \) with the properties A1-A2 for a general member \( D' \in |\mathcal{O}_R(D)| \), there exist a dominant and generically finite map \( \rho : U \to \mathbb{P}^N \).
$|O_R(D)|$ and a smooth projective variety $Q$ with the commutative diagram

$$
(2.43) \quad \xymatrix{ Q \ar[r]^\varphi & R \\
(D \times \rho U) \ar[r] \ar[u]^{\sim} & (D \times \rho U) \times \mathbb{P}^1 \ar[u]^f }
$$

where $\varphi$ maps a general fiber $Q$ of $\pi_U \circ f : Q \to D \times \rho U \to U$ to $R$ with the properties A1-A2 and the existence of a birational map between $(D \times \rho U) \times \mathbb{P}^1$ and $Q$ is due to the fact that the fiber of $\pi_U \circ f$ over a general point $u \in U$ is a $\mathbb{P}^1$-bundle $Q_u$ over $D_u := \pi_U^{-1}(u) \in |O_R(D)|$. Here we use the notations $\pi_D$ and $\pi_U$ for the projections $D \times \rho U \to D$ and $D \times \rho U \to U$, respectively.

Let $p = (D', p)$ be a general point on $D$. The pre-image $(\pi_D \circ f)^{-1}(p)$ consists of $N$ copies of $\mathbb{P}^1$, say, $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$, where $N = \deg p$. That is,

$$
(2.44) \quad (\pi_D \circ f)^* p = \Gamma_1 + \Gamma_2 + \ldots + \Gamma_N.
$$

For each $\Gamma_k$, we have

$$
(2.45) \quad \varphi_* \Gamma_k \cdot D' = (\alpha_i + \beta_{i-1})p + \Sigma_k
$$

with $\Sigma_k$ supported on $Z$, where $Z$ is a subvariety of $D'$ of codimension $\geq 1$, independent of $p$. Therefore,

$$
(2.46) \quad \varphi_*((\pi_D \circ f)^* p) \cdot D' = N(\alpha_i + \beta_{i-1})p + \Sigma
$$

with $\Sigma$ supported on $Z$. By (2.42), we still have a surjection (2.39) on $D'$. Since the above argument works for a general member $D' \in |O_R(D)|$, it holds for $D$ and we are done.

**References**


632 Central Academic Building, University of Alberta, Edmonton, Alberta T6G 2G1, Canada
E-mail address: xichen@math.ualberta.ca