THE REAL REGULATOR FOR A SELF-PRODUCT OF A GENERAL SURFACE

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ABSTRACT. In [C-L3] it is shown that the real regulator for a general self-product of a $K3$ surface is nontrivial. In this note, we prove a theorem which says that the real regulator for a general self-product of a surface of higher order (in a suitable sense), is essentially trivial.

1. STATEMENT OF THE THEOREM

Let Γ be a smooth projective curve, $\{Z_t\}_{t\in\Gamma}$ a family of surfaces, all defined over a subfield $k \subset \mathbb{C}$, and put:

$$Z_\Gamma := \coprod_{t\in\Gamma} Z_t.$$

Let us assume that Z_Γ is smooth and that any singular Z_t is nodal. In local analytic coordinates the singular set of $Z_\Gamma \times_\Gamma Z_\Gamma$ looks like

$$x_1^2 + x_2^2 + x_3^2 = t^M = y_1^2 + y_2^2 + y_3^2,$$

for some positive integer M. Since we assume Z_Γ to be smooth, we necessarily have $M = 1$. Then locally we are in the situation of

$$x_1^2 + x_2^2 + x_3^2 - (y_1^2 + y_2^2 + y_3^2) = 0,$$

which is an isolated nodal singularity. Then the projectivized tangent cone is a 4-dimensional smooth quadric Q_0. Let $[Z_\Gamma \times_\Gamma Z_\Gamma]_0$ be the blow-up of $Z_\Gamma \times_\Gamma Z_\Gamma$ at this isolated singular point 0. We are going to assume that $\mathrm{CH}^2(Q_0, \mathbb{Q}) \hookrightarrow \mathrm{CH}^3([Z_\Gamma \times_\Gamma Z_\Gamma]_0, \mathbb{Q})$ is injective. Further, we assume that $\mathrm{CH}^1(Z_t; \mathbb{Q}) := \mathrm{CH}^1(Z_t) \otimes \mathbb{Q} \simeq \mathbb{Q}$ for all $t \in \Gamma$. Note that this (latter) condition alone will fail for a general 1-parameter

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family of quartic surfaces in $\mathbb{P}^3$ (take for example the locus of quartics containing a line). Let us further assume that for all $t \in \Gamma$:

$$\begin{aligned} \mathrm{CH}^2(Z_t \times Z_t;\mathbb{Q}) = \ & \mathrm{CH}^0(Z_t;\mathbb{Q}) \otimes \mathrm{CH}^2(Z_t;\mathbb{Q}) \\ & + \mathrm{CH}^1(Z_t;\mathbb{Q}) \otimes \mathrm{CH}^1(Z_t;\mathbb{Q}) \\ & + \mathrm{CH}^2(Z_t;\mathbb{Q}) \otimes \mathrm{CH}^0(Z_t;\mathbb{Q}) + \mathbb{Q}\{\Delta_{Z_t}\}, \end{aligned} \tag{1.1}$$

where Δ_{Z_t} is the diagonal image $Z_t \to Z_t \times Z_t$. This is not an unreasonable assumption, given the fact that a similar kind of decomposition (specifically without the $\mathbb{Q}\{\Delta_{Z_t}\}$ term), holds for a general product of $K3$ (and higher order) surfaces, under the assumption of a rather deep conjecture (due to Bloch and Beilinson) - see[L] and [C-L2].

Let $B \subset \Gamma$ be a finite set for which $Z_U \times_U Z_U \to U$ is smooth and proper, where $U = \Gamma\backslash B$. Finally, for a very general choice of $t \in \Gamma$, assume that with respect to the monodromy representation

$$\pi_1(U) \to \mathrm{Aut}\left(H^4_{\mathrm{tr}}(Z_t \times Z_t, \mathbb{Q})\right),$$

there are no classes in the transcendental cohomology $H^4_{\mathrm{tr}}(Z_t \times Z_t, \mathbb{C})$ whose Hodge (p, q) components displace horizontally with respect to the Gauss-Manin connection (the reader may wish to consult [C-L2] for a precise definition of this).

Theorem 1.2. *Given the above setting, assume that $c_2(Z_t) \neq 3$ for $t \in U$. Let $t \in \Gamma(\mathbb{C})$ correspond to an embedding $k(\Gamma) \hookrightarrow \mathbb{C}$. Then the reduced regulator map*

$$\underline{r}_{3,1} : \mathrm{CH}^3(Z_{k(\Gamma)} \times Z_{k(\Gamma)}, 1) \to H^4_{\mathrm{tr}}(Z_t \times Z_t, \mathbb{R}) \bigcap H^{2,2}(Z_t \times Z_t),$$

is zero.

Note that for a general self-product of a $K3$ surface, the reduced regulator is nontrivial ([C-L3]). What the theorem says is that for a self-product of a general surface of higher order, the reduced regulator is trivial. If we consider for example surfaces in $\mathbb{P}^3$, then in light of the fact that a smooth surface in $\mathbb{P}^3$ is $K3 \Leftrightarrow$ its degree $d = 4$, a higher order surface in this context should be a general surface in $\mathbb{P}^3$ of degree ≥ 5. Theorem 1.2 however, does not directly apply to surfaces in $\mathbb{P}^3$ of degree $d \geq 5$. The subtle point here is that a Lefschetz pencil of surfaces, after an arbitrary base change, is no longer smooth, while Z_Γ is assumed to be smooth in Theorem 1.2. However the real issue is that the injectivity statement $\mathrm{CH}^2(Q_0, \mathbb{Q}) \hookrightarrow \mathrm{CH}^3([Z_\Gamma \times_\Gamma Z_\Gamma]_0, \mathbb{Q})$ needs to be addressed. But this can be fixed at least for surfaces in $\mathbb{P}^3$. Namely, we have the following.

Theorem 1.3. *For a very general surface $Z_t \subset \mathbb{P}^3$ of degree $d \geq 5$, the reduced regulator map*

$$\underline{r}_{3,1} : \mathrm{CH}^3(Z_t \times Z_t, 1) \to H^4_{\mathrm{tr}}(Z_t \times Z_t, \mathbb{R}) \bigcap H^{2,2}(Z_t \times Z_t),$$

is zero if we assume that (1.1) *holds for all $t \in \mathbb{P}^N \backslash \Sigma$, where $\mathbb{P}^N$ is the parameter space of surfaces of degree d in $\mathbb{P}^3$ and $\Sigma \subset \mathbb{P}^N$ is a countable union of subvarieties of $\mathbb{P}^N$ with codimension ≥ 2.*

Implicit in the statement of this theorem is the expectation that a very general surface $Z_t \subset \mathbb{P}^3$ of degree $d \geq 5$ automatically satisfies the assumption that $\mathrm{codim}_{\mathbb{P}^N}\Sigma \geq 2$. There are good reasons to expect this. Firstly, if one assumes a conjecture of Bloch and Beilinson on the injectivity of the Abel-Jacobi map for smooth quasiprojective varieties defined over number fields, then according to [C-L2] and [L], such a decomposition in (1.1) will hold for Z_t replaced by a very general X. Further, apart from a number of technical issues, the key issue is requiring (1.1) to hold for all $t \in \Gamma$. Such an X will be a very general member of a general (Lefschetz) pencil $\{Z_t\}_{t\in\mathbb{P}^1}$ of surfaces of degree d in $\mathbb{P}^3$. As explained in [C-L1], the work of M. Green ([G]) on the Noether-Lefschetz locus implies that $\mathrm{CH}^1(Z_t;\mathbb{Q}) \cong \mathbb{Q}$, for all $t \in \mathbb{P}^1$, provided that $d \geq 5$. Although the proof in [G] relies on infinitesimal Hodge theoretic methods, an ad hoc explanation goes as follows. The horizontal displacement of a rational topological 2-cycle in the Noether-Lefschetz locus must pair to zero under integration with holomorphic 2-forms. That $d \geq 5 \Rightarrow \dim H^{2,0}(X) > 1$ suggests that this locus is of codimension ≥ 2 in the universal family of surfaces of degree $d \geq 5$ in $\mathbb{P}^3$ (which is indeed a fact). This very same reasoning suggests the decomposition in (1.1), which holds for very general t under our above assumptions, actually holds for all $t \in \mathbb{P}^1$. Finally, one can show that $\deg(c_2(X)) = d(d^2 + 6 - 4d)$.

The ideas presented here are similar to those given in [C-L2]. Rather than repeat them, we highlight the main points and introduce the new ingredients.

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2. Proof of Theorem 1.2

This will be carried out in three steps.

Step I: *The spread cycle* ξ. Put

$$Z_B = \coprod_{t\in B} Z_t \times Z_t \subset Z_\Gamma \times_\Gamma Z_\Gamma.$$

Let $\xi_{k(\Gamma)} \in \mathrm{CH}^3(Z_{k(\Gamma)} \times Z_{k(\Gamma)}, 1)$. After possibly enlarging B, we may assume that:

(i) $\xi_{k(\Gamma)}$ spreads to a cycle $\xi \in \mathrm{CH}^3\left(Z_\Gamma \times_\Gamma Z_\Gamma \backslash Z_B, 1\right)$,

Further, up to a base change $\Gamma' \twoheadrightarrow \Gamma$, we can assume that:

(ii) There is a section $\sigma : \Gamma \to Z_\Gamma$ avoiding the double points of the singular fibers, with $\Gamma \simeq \sigma(\Gamma)$. (Note: Our goal is to complete ξ to a cycle on $Z_{\Gamma'} \times_{\Gamma'} Z_{\Gamma'}$. Once we do that, then we can proper push-forward it to $Z_\Gamma \times_\Gamma Z_\Gamma$. Therefore we may assume for simplicity that $\Gamma = \Gamma'$.) For $t \in U$, let $\xi_t \in \mathrm{CH}^3(Z_t \times Z_t, 1)$ be the corresponding class.

We will refer to the diagram

$$\begin{array}{ccccc} Z_\Gamma & & & & \\ & \searrow^{\Delta} & & & \\ & & Z_\Gamma \times_\Gamma Z_\Gamma & \xrightarrow{Pr_1} & Z_\Gamma \\ & & \downarrow{\scriptstyle Pr_2} & & \downarrow \\ & & Z_\Gamma & \longrightarrow & \Gamma \end{array}$$

Step II: *Modifying* ξ *in such a way that it extends to a cycle* $\underline{\xi} \in \mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma, 1)$*, and such that* $\underline{r}_{3,1}(\xi_t) = \underline{r}_{3,1}(\underline{\xi}_t)$ *for general* $t \in \Gamma$. The closure of ξ defines a precycle $\bar{\xi}$ on $Z_\Gamma \times_\Gamma Z_\Gamma$ whose boundary $\partial\bar{\xi}$ is supported on Z_B. Thus according to the decomposition in (1.1), we have

$$\{\partial\bar{\xi}\} = R + S + T + W = 0 \in \mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma; \mathbb{Q}),$$

where

$$R = \sum_{t\in B} Z_t \otimes \xi_t^{(R)}, \qquad S = \sum_{t\in B} n_t H_t \otimes H_t'$$
$$T = \sum_{t\in B} \xi_t^{(T)} \otimes Z_t, \qquad W = \sum_{t\in B} m_t \Delta_{Z_t}$$

Here H_t and H'_t are general hyperplane sections of Z_t, $\xi_t^{(R)}$, $\xi_t^{(T)}$ are 0-cycles, and $n_t,\ m_t \in \mathbb{Q}$. Now put

$$\begin{aligned} D_1 &= \coprod_{t\in\Gamma} \sigma(t) \times Z_t \quad \simeq Z_\Gamma \\ D_2 &= \coprod_{t\in\Gamma} Z_t \times \sigma(t) \quad \simeq Z_\Gamma \\ H &= \coprod_{t\in\Gamma} H'_t \otimes H_t \\ \Delta &= \Delta(Z_\Gamma) \end{aligned}$$

Let $d = \deg Z_t = \left(H_t^2\right)_{Z_t}$, and put $\hat{\xi} = R + S + T + W$. Also put $N = \left(\Delta_{Z_t}^2\right)_{Z_t\times Z_t}$ for any $t \in U$. Of course, this number is really independent of $t \in \Gamma$, by defining it as a limit $t \mapsto t_0$ over $t \in U$, for any $t_0 \in \Gamma$; and observe that for $t \in U$, $N = \deg(c_2(Z_t))$.

We compute:

$$\begin{aligned} \hat{\xi} \cap D_1 &= \sum_{t\in B} \sigma(t) \times \xi_t^{(R)} + \sum_{t\in B} m_t(\sigma(t), \sigma(t)) \sim_{\text{rat}} 0 \text{ on } D_1 \\ \hat{\xi} \cap D_2 &= \sum_{t\in B} \xi_t^{(T)} \times \sigma(t) + \sum_{t\in B} m_t(\sigma(t), \sigma(t)) \sim_{\text{rat}} 0 \text{ on } D_2 \end{aligned}$$

These equations yield:

$$\begin{aligned} \sum_{t\in B} \Big(\big[\deg(\xi_t^{(R)}) \big] \cdot t + m_t \cdot t \Big) \sim_{\text{rat}} 0 \text{ on } \Gamma \\ \sum_{t\in B} \Big(\big[\deg(\xi_t^{(T)}) \big] \cdot t + m_t \cdot t \Big) \sim_{\text{rat}} 0 \text{ on } \Gamma \end{aligned}$$

Further,

$$\begin{aligned} \hat{\xi} \cap H \sim_{\text{rat}} 0 &\Rightarrow d^2\big(\sum_{t\in B} n_t \cdot t\big) + d \sum_{t\in B} m_t \cdot t \sim_{\text{rat}} 0 \text{ on } \Gamma \\ &\Rightarrow d\big(\sum_{t\in B} n_t \cdot t\big) + \sum_{t\in B} m_t \cdot t \sim_{\text{rat}} 0 \text{ on } \Gamma \\ \hat{\xi} \cap \Delta \sim_{\text{rat}} 0 &\Rightarrow d\big(\sum_{t\in B} n_t \cdot t\big) + N \sum_{t\in B} m_t \cdot t \\ &\qquad + \sum_{t\in B} \big[\deg(\xi_t^{(R)}) \big] \cdot t + \sum_{t\in B} \big[\deg(\xi_t^{(T)}) \big] \cdot t \sim_{\text{rat}} 0 \text{ on } \Gamma \end{aligned}$$

Now put

$$x_1 = \sum_{t\in B}[\deg(\xi_t^{(R)})]\cdot t$$

$$x_2 = \sum_{t\in B}[\deg(\xi_t^{(T)})]\cdot t$$

$$x_3 = \sum_{t\in B} m_t\cdot t$$

$$x_4 = \sum_{t\in B} n_t\cdot t$$

Then in terms of rational equivalence to zero on Γ, we have

$$\begin{bmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & d\\ 1 & 1 & N & d\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\\ 0\end{bmatrix}$$

A simple computation yields

$$\det\begin{bmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & d\\ 1 & 1 & N & d\end{bmatrix} = d(3-N),$$

which is nonzero by our assumptions. In particular $x_j \sim_{\rm rat} 0$ on Γ. As in [C-L2], by multiplying H and Δ by the relevant rational functions pulled back from Γ, one can then easily modify $\overline{\xi}$ so that $S = W = 0$. On D_1,

$$\sum_{t\in B}\sigma(t)\times\xi_t^{(R)} \sim_{\rm rat} 0,$$

and on D_2,

$$\sum_{t\in B}\xi_t^{(T)}\times\sigma(t) \sim_{\rm rat} 0.$$

Each of these involves rational functions on curves $C \subset D_i$ where either C dominates Γ or $C\subset Z_t$ for some $t\in\Gamma$. Via the projections $Pr_j : Z_\Gamma\times_\Gamma Z_\Gamma \to Z_\Gamma \simeq D_j$, this all lifts to $R\sim_{\rm rat} 0$ and $T\sim_{\rm rat} 0$ on $Z_\Gamma\times_\Gamma Z_\Gamma$. Again, by a further modification of $\overline{\xi}$ we arrive at a sought for class $\underline{\xi}\in {\rm CH}^3(Z_\Gamma\times_\Gamma Z_\Gamma, 1)$, such that $\underline{r}_{3,1}(\xi_t) = \underline{r}_{3,1}(\underline{\xi}_t)$ for general $t\in\Gamma$, where we reiterate that $\underline{r}_{3,1}(\underline{\xi}_t)$ is a projection of the real regulator image which kills the image of the decomposable cycles, (and for which we modified by cycles which are fiberwise decomposable).

<u>Step III</u>: *A rigidity argument.* The variety $Z_\Gamma\times_\Gamma Z_\Gamma$ is singular (unless $Z_\Gamma \twoheadrightarrow \Gamma$ is smooth) , and so we observe that there is a cycle map on

the level of homology:

$$\mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma, 1; \mathbb{Q}) \to H_5^{\mathcal{D}}(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(2)),$$

where $H_5^{\mathcal{D}}(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(2))$ is real Deligne homology, and $\mathbb{R}(m)$ is the Tate twist. There is an exact sequence $H_6(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(3)) \to H_5^{\mathcal{D}}(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(2)) \twoheadrightarrow \hom_{\mathbb{R}-\mathrm{MHS}}\big(\mathbb{R}(0), H_5(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(2))\big)$. We will show (below) that modulo the addition of a decomposable class arising from $\mathrm{CH}^3(Z_{k(\Gamma)} \times Z_{k(\Gamma)}, 1; \mathbb{Q})$, $\underline{\xi} \in \mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma, 1; \mathbb{Q})$ has zero image in $\hom_{\mathbb{R}-\mathrm{MHS}}\big(\mathbb{R}(0), H_5(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(2))\big)$. Hence we can assume that $\underline{\xi} \in H_5^{\mathcal{D}}(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(2))$ lifts to a class in $H_6(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(3))$. Now consider the composite

$$H_6(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(3)) \to H_4(Z_t \times Z_t, \mathbb{R}(2)) \xrightarrow{\sim} H^4(Z_t \times Z_t, \mathbb{R}(2)) \twoheadrightarrow H^{2,2}(Z_t \times Z_t, \mathbb{R}(2)).$$

Then as in [C-L1], we can apply Betti rigidity, together with our monodromy assumptions to deduce that $\underline{r}_{3,1}(\underline{\xi}_t)$ (a fortiori $\underline{r}_{3,1}(\xi_t)$) is zero for sufficiently general $t \in \Gamma$. We now attend to the details of modifying $\underline{\xi}$. In local analytic coordinates the singular set of $Z_\Gamma \times_\Gamma Z_\Gamma$ looks like

$$x_1^2 + x_2^2 + x_3^2 = t^M = y_1^2 + y_2^2 + y_3^2,$$

for some positive integer M. Since we assume Z_Γ to be smooth, we necessarily have $M = 1$. Then locally we are in the situation of

$$x_1^2 + x_2^2 + x_3^2 - (y_1^2 + y_2^2 + y_3^2) = 0,$$

which is an isolated nodal singularity. Then the projectivized tangent cone is a 4-dimensional smooth quadric Q_0 whose generators contribute to $\hom_{\mathbb{R}-\mathrm{MHS}}\big(\mathbb{R}(0), Gr_W^0 H_5(Z_\Gamma \times_\Gamma Z_\Gamma, \mathbb{R}(2))\big)$. Let $[\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}]_0$ be the blow-up of $Z_\Gamma \times_\Gamma Z_\Gamma$ at this isolated singular point 0. Now recall by assumption that we have an injection $\mathrm{CH}^2(Q_0; \mathbb{Q}) \hookrightarrow \mathrm{CH}^3([\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}]_0; \mathbb{Q})$. From the localization sequence

$$\cdots \to \mathrm{CH}^3([\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}]_0, 1; \mathbb{Q}) \to \mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma \backslash \{0\}, 1; \mathbb{Q}) \to \mathrm{CH}^2(Q_0; \mathbb{Q}) \hookrightarrow \mathrm{CH}^3([\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}]_0; \mathbb{Q}) \to \cdots,$$

it is clear that $\underline{\xi}$ lifts to a class in $\mathrm{CH}^3([\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}]_0, 1; \mathbb{Q})$. Repeating this procedure for each nodal singularity, we arrive at $\underline{\xi}$ lying in the image of

$$\mathrm{CH}^3(\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}, 1; \mathbb{Q}) \to \mathrm{CH}^3(Z_\Gamma \times_\Gamma Z_\Gamma, 1; \mathbb{Q}),$$

where $\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}$ is a desingularization of $Z_\Gamma \times_\Gamma Z_\Gamma$. Now use the fact that $\hom_{\mathbb{R}-\mathrm{MHS}}\big(\mathbb{R}(0), H_5(\widetilde{Z_\Gamma \times_\Gamma Z_\Gamma}, \mathbb{R}(2))\big) = 0$. □

Remark 2.1. Let us put

$$\widetilde{Z}_t = \begin{cases} Z_t & \text{if } Z_t \text{ smooth} \\ \text{desing}(Z_t) & \text{if } Z_t \text{ singular} \end{cases}$$

where $\text{desing}(Z_t)$ is the minimal desingularization of Z_t. Suppose that

$$H^{2,2}(\widetilde{Z}_t \times \widetilde{Z}_t) \bigcap H^2_{\text{tr}}(\widetilde{Z}_t, \mathbb{Q}) \otimes H^2_{\text{tr}}(\widetilde{Z}_t, \mathbb{Q}) \simeq \mathbb{Q},$$

and that

$$\text{CH}^1(Z_t; \mathbb{Q}) \simeq \mathbb{Q},$$

for all $t \in \Gamma$. (This last condition implies that $H^1(\widetilde{Z}_t, \mathbb{Q}) = 0 = H^3(\widetilde{Z}_t, \mathbb{Q})$.) Then assuming the existence of the conjectured Bloch-Beilinson filtration, one can show as in [C-L2] that the decomposition in (1.1) holds.

3. Proof of Theorem 1.3

Let $Z_{\mathbb{P}^1} \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a Lefschetz pencil of surfaces of degree d in $\mathbb{P}^3$ with $\mathbb{P}^1 \subset \mathbb{P}^N$. For a very general choice of $\mathbb{P}^1 \subset \mathbb{P}^N$, we can make it avoid Σ and the locus where $\text{Pic}(Z_t) \neq \mathbb{Z}$. Suppose that we have a class $\xi_t \in \text{CH}^3(Z_t \times Z_t, 1; \mathbb{Q})$ for $t \in \mathbb{P}^1$ general. Then after a base change $\Gamma \to \mathbb{P}^1$, we can extend ξ_t to ξ_U over an open set $U \subset \Gamma$. By the same argument in the previous section, we can further extend ξ to all of $Z_\Gamma \times_\Gamma Z_\Gamma$, where $Z_\Gamma = Z_{\mathbb{P}^1} \times_{\mathbb{P}^1} \Gamma$. As mentioned at very beginning, Z_Γ might be singular. Hence, in order to show the vanishing of $\underline{r}_{3,1}(\xi)$ by the monodromy argument, we need to lift ξ to a desingularization of $Z_\Gamma \times_\Gamma Z_\Gamma$.

We first desingularize Z_Γ. Let Y be the minimal desingularization of Z_Γ. Observe that the singularities of Z_Γ consist of the points $p \in Z_t$, where p is an ordinary double point of the surface Z_t and the finite map $\Gamma \to \mathbb{P}^1$ is ramified at $t \in \Gamma$. Locally at p, Z_Γ is given by

$$x_1^2 + x_2^2 + x_3^2 = t^M \tag{3.1}$$

where M is ramification index $\Gamma \to \mathbb{P}^1$ at t. For simplicity, we may assume that M is even; otherwise, we just replace Γ by Γ' with a further base change $\Gamma' \to \Gamma$. The singularity p as in (3.1) can be resolved by a sequence of blowups and we end up with

$$Y_t = \widetilde{Z}_t \cup Q_1 \cup Q_2 \cup \ldots \cup Q_m$$

locally over p, where $Q_0 = \widetilde{Z}_t$ is the proper transform of Z_t and $Q_1, Q_2, \ldots, Q_m$ are a chain of $m = M/2$ rational ruled surfaces satisfying that

- $Q_i \cong \mathbb{F}_2$ for $1 \leq i \leq m-1$ and $Q_m \cong \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$;

- $Q_i \cap Q_j \neq \emptyset$ iff $|i-j| \leq 1$;
- Q_{i-1} and Q_i meet transversely along a curve $C_i \cong \mathbb{P}^1$ for $i = 1, 2, ..., m$.

Let us consider $Y \times_\Gamma Y$. This is only a partial resolution of $Z_\Gamma \times_\Gamma Z_\Gamma$; $Y \times_\Gamma Y$ is singular along $C_i \times C_j$ with local equation $x_1 x_2 = y_1 y_2 = t$. On the other hand, $Y \times_\Gamma Y$ admits a small resolution $\widetilde{Y \times_\Gamma Y}$ and we can lift every class in $\mathrm{CH}^3(Y \times_\Gamma Y, 1; \mathbb{Q})$ to $\mathrm{CH}^3(\widetilde{Y \times_\Gamma Y}, 1; \mathbb{Q})$. This small resolution is obtained by subsequently blowing up $Q_i \times Q_j$; locally we are blowing up along $x_1 = y_1 = 0$. Note that $\widetilde{Y \times_\Gamma Y}$ is projective as it is obtained by blowing up along algebraic subvarieties. The exceptional loci of this resolution consist of threefolds E_{ij} which are $\mathbb{P}^1$ bundles over $C_i \times C_j$ for $1 \leq i, j \leq m$. So the obstruction to the lifting comes from the map

$$\bigoplus_{i,j} \mathrm{CH}^1(E_{ij}; \mathbb{Q}) \to \mathrm{CH}^3(\widetilde{Y \times_\Gamma Y}; \mathbb{Q}) \tag{3.2}$$

by the corresponding localization sequence; a class in $\mathrm{CH}^3(Y \times_\Gamma Y, 1; \mathbb{Q})$ can be lifted to $\mathrm{CH}^3(\widetilde{Y \times_\Gamma Y}, 1; \mathbb{Q})$ if the map (3.2) is injective.

Let $\widetilde{Q_i \times Q_j} \subset \widetilde{Y \times_\Gamma Y}$ be the proper transform $Q_i \times Q_j$. The 3-fold E_{ij} is the intersection of two out of the four 4-folds among

$$\{\widetilde{Q_{i-\alpha} \times Q_{j-\beta}} : \alpha, \beta = 0 \text{ or } 1\}$$

and meets the other two transversely along two disjoint sections of $E_{ij} \to C_i \times C_j$, say G_{ij} and $G'_{ij} \subset E_{ij}$; exactly which two depends on the order in which we blow up $Q_i \times Q_j$. Obviously,

$$\mathrm{CH}^1(E_{ij}) \cong \mathrm{CH}^1(C_i \times C_j) \oplus \mathbb{Z} G_{ij}. \tag{3.3}$$

On Y, we obviously have the injection

$$\bigoplus_i \mathrm{CH}^k(C_i) \hookrightarrow \mathrm{CH}^{k+2}(Y) \tag{3.4}$$

for $k = 0, 1$. This gives us the injection

$$\bigoplus_{i,j} \mathrm{CH}^1(C_i \times C_j) \hookrightarrow \mathrm{CH}^3(\widetilde{Y \times_\Gamma Y}). \tag{3.5}$$

On the other hand, it is easy to show the injection

$$\bigoplus_{i,j} (\mathbb{Z} G_{ij} \oplus \mathbb{Z} G'_{ij}) \hookrightarrow \mathrm{CH}^3(\widetilde{Y \times_\Gamma Y}) \tag{3.6}$$

and the injectivity of (3.2) follows.

So it remains to lift classes in $\mathrm{CH}^3(Z \times_\Gamma Z, 1; \mathbb{Q})$ to $\mathrm{CH}^3(Y \times_\Gamma Y, 1; \mathbb{Q})$. The obstruction is the map

$$\bigoplus_{i,j} \mathrm{CH}^2(Q_i \times Q_j; \mathbb{Q}) \to \mathrm{CH}^3(Y \times_\Gamma Y; \mathbb{Q}) \tag{3.7}$$

and it suffices to show that (3.7) is injective.

Obviously,

$$\mathrm{CH}^2(Q_i \times Q_j) = \bigoplus_{k=0}^{2} \mathrm{CH}^k(Q_i) \otimes \mathrm{CH}^{2-k}(Q_j) \tag{3.8}$$

and the injections

$$\mathrm{CH}^k(Q_i) \hookrightarrow \mathrm{CH}^{k+1}(Y) \tag{3.9}$$

are trivial for $k = 0, 2$. The hard part is to prove (3.9) for $k = 1$, i.e.,

$$\bigoplus_{i} \mathrm{CH}^1(Q_i) \hookrightarrow \mathrm{CH}^2(Y). \tag{3.10}$$

This has been done in the appendix of [C-L2], where it was proved that (3.10) is an injection if $Z_{\mathbb{P}^1}$ is chosen to be a very general pencil.

References

[B] S. Bloch, Algebraic cycles and higher K-theory, *Advances in Math.* **61** (1986), 267-304.

[C-L1] X. Chen and J. D. Lewis, Noether-Lefschetz for K_1 of a certain class of surfaces, *Bol. Soc. Mat. Mexicana* **10(3)** (2004), 29-41.

[C-L2] ———, The real regulator for a product of $K3$ surfaces, in *Mirror Symmetry V, Proceedings of the BIRS conference in Banff, Alberta* (Edited by S.-T. Yau, N. Yui and J. D. Lewis), AMS/IP Studies in Advanced Mathematics, Volume **38**. (2006), 271-283.

[C-L3] ———, Real Regulators on Self-products of $K3$ Surfaces. To appear in the J. of Algebraic Geometry.

[C-L4] ———, The Hodge-$\mathcal{D}$-conjecture for $K3$ and Abelian surfaces, *J. Alg. Geom.* **14** (2005), 213-240.

[G] M. Green, A new proof of the explicit Noether-Lefschetz theorem, *J. Differential Geometry* **27** (1988), 155-159.

[L] J. D. Lewis, A filtration on the Chow groups of a complex projective variety, *Compositio Math.* **128**, (2001), 299-322.

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