Bol. Soc. Mat. Mexicana (3) Vol. 10, Special issue, 2004

# NOETHER-LEFSCHETZ FOR $K_1$ OF A CERTAIN CLASS OF SURFACES

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Abstract. We first give an elementary new proof of the vanishing of the regulator on $K_1(Z)$ where $Z \subset \mathbb{P}^3$ be a general surface of degree $d \geq 5$, using a Lefschetz pencil argument. By a similar argument we then show the triviality of the regulator for $K_1$ of a general product of two curves.

## 1. Statement of results.

Let $Z$ be a smooth quasiprojective variety over $\mathbb{C}$, and for given nonnegative integers $k, m$, let $\mathrm{CH}^k(Z, m)$ be the higher Chow group as introduced in [Blo1]. In [Blo2], Bloch constructs a cycle class map into any suitable cohomology theory. In our setting, the corresponding map is:

$$\mathrm{cl}_{k,m}\colon\ \mathrm{CH}^k(Z,m) \to H^{2k-m}_{\mathcal{D}}(Z, \mathbb{Q}(k)),$$

where $H^{2k-m}_{\mathcal{D}}(Z, \mathbb{Q}(k))$ is Deligne-Beilinson cohomology, which fits in a short exact sequence

$$0 \to \frac{H^{2k-m-1}(Z,\mathbb{C})}{F^kH^{2k-m-1}(Z,\mathbb{C}) + H^{2k-m-1}(Z,\mathbb{Q}(k))} \to H^{2k-m}_{\mathcal{D}}(Z,\mathbb{Q}(k))$$

$$\to F^kH^{2k-m}(Z,\mathbb{C}) \bigcap H^{2k-m}(Z,\mathbb{Q}(k)) \to 0.$$

Our primary interest is when $Z$ is also complete, and $m = 1$. Thus one has the corresponding map:

$$\mathrm{cl}_{k,1}\colon\ \mathrm{CH}^k(Z,1) \to \frac{H^{2k-2}(Z,\mathbb{C})}{F^kH^{2k-2}(Z,\mathbb{C}) + H^{2k-2}(Z,\mathbb{Q}(k))}.$$

Let $\mathrm{Hg}^{k-1}(Z) := H^{2k-2}(Z, \mathbb{Q}(k-1)) \cap F^{k-1}H^{2k-2}(Z, \mathbb{C})$ be the Hodge group. Then one has an induced map

$$\underline{\mathrm{cl}}_{k,1}\colon\ \mathrm{CH}^k(Z,1) \to \frac{H^{2k-2}(Z,\mathbb{C})}{F^kH^{2k-2}(Z,\mathbb{C}) + \mathrm{Hg}^{k-1}(Z)\otimes\mathbb{C} + H^{2k-2}(Z,\mathbb{Q}(k))}.$$

It is known that $\underline{\mathrm{cl}}_{k,1}$ is trivial for $Z$ a sufficiently general complete intersection and of sufficiently high multidegree. This is an consequence of the work of Nori [No], together with a technique similar to that given in [G-S]. The argument is

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2000 *Mathematics Subject Classification*: 14C25, 14C30, 14C35.

*Keywords and phrases*: Regulator, Deligne cohomology, Chow group.

First author partially supported by a startup grant from the University of Alberta. Second author partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

presented in [MS]. Further, it is noted in [MS], based on an effective bound in [Pa], that

$$\underline{\mathrm{cl}}_{2,1}\colon\ \mathrm{CH}^2(Z,1) \to \frac{H^2(Z,\mathbb{C})}{F^2H^2(Z,\mathbb{C}) + \mathrm{Hg}^1(Z)\otimes\mathbb{C} + H^2(Z,\mathbb{Q}(2))},$$

is trivial for sufficiently general surfaces $Z \subset \mathbb{P}^3$ of degree $d \geq 5$. The method of Nori involves passing to the universal family of complete intersections of a given multidegree, in a given projective space. A similar point of view appears in [Na]. In this paper, we give an elementary and direct proof of the triviality of $\underline{\mathrm{cl}}_{2,1}$ for a general surface $Z \subset \mathbb{P}^3$ of degree $\geq 5$, by working with a Lefschetz pencil of degree $d \geq 5$ surfaces in $\mathbb{P}^3$. Thus our first main result is an elementary new proof of the following:

Theorem (1.1). *For a sufficiently general surface $Z \subset \mathbb{P}^3$ of degree $d \geq 5$, the map $\underline{\mathrm{cl}}_{2,1}$ is trivial.*

We remark that the theorem is trivially true, without the generic hypothesis, if $\deg Z \leq 3$, as $H^2(Z)$ is algebraic. From the works of Collino, Voisin, S. Müller-Stach, *et al,* and more recently the authors [C-L], it is false if $\deg Z = 4$. Since our method requires only a Lefschetz pencil as opposed to the universal family of surfaces of degree $d$ in $\mathbb{P}^3$, and that it provides a rather simple proof of a counterexample of the Hodge-$\mathcal{D}$-conjecture of Beilinson [Bei1], we believe that this approach has some merit. In particular, a variant of this argument leads to our next result:

Theorem (1.2). *Let $X = C_1 \times C_2$ be a product of two general curves (resp. general hyperelliptic curves), where the genus $g(C_1) \geq 1$ and $g(C_2) \geq 2$. Then $\underline{\mathrm{cl}}_{2,1}$ is trivial.*

We remark that story is *false* if $C_1$ and $C_2$ are general elliptic curves ([C-L]).

*Question* (1.3). Consider a smooth projective surface $X$ and the corresponding regulators

$$r_{2,1}\colon\ \mathrm{CH}^2(X,1;\mathbb{R}) \to H^{1,1}(X,\mathbb{R}(1)),$$

$$\underline{\mathrm{cl}}_{2,1}\colon\ \mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{R}) \to \frac{H^3_{\mathcal{D}}(X,\mathbb{Q}(1))}{\mathrm{Hg}^1(X)\otimes\mathbb{C}/\mathbb{Q}(1)}.$$

Let $\kappa(X)$ be the Kodaira dimension. If $\kappa(X) = -1$, then $r_{2,1}$ is surjective. If $\kappa(X) = 0$, then $r_{2,1}$ is surjective for "general" $X$ (see [C-L]). If $X$ is "general" and if $\kappa(X) \geq 1$, is it the case that $\underline{\mathrm{cl}}_{2,1}$ is trivial?

## 2. Some definitions

(1) *Deligne cohomology.* A good source for the definition of Deligne cohomology can be found in [EV]. For our narrow purposes, the following will suffice. Let $X$ be a projective algebraic manifold, and for a subring $\mathbb{A} \subset \mathbb{R}$, put $\mathbb{A}(j) = \mathbb{A}(2\pi\sqrt{-1})^j$. Consider the Deligne complex

$$\mathbb{A}(j)_{\mathcal{D}}\colon\quad \mathbb{A}(j) \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{j-1}_X.$$

*Definition* (2.1). Deligne cohomology is given by $H^i_{\mathcal{D}}(X,\mathbb{A}(j)) := \mathbb{H}^i(\mathbb{A}(j)_{\mathcal{D}})$ (hypercohomology).

One has a short exact sequence

$$0 \to \frac{H^{i-1}(X,\mathbb{C})}{F^j H^{i-1}(X,\mathbb{C}) + H^{i-1}(X,\mathbb{A}(j))} \to H^i_{\mathcal{D}}(X,\mathbb{A}(j))$$
$$\to F^j H^i(X,\mathbb{C}) \bigcap H^i(X,\mathbb{A}(j)) \to 0.$$

We are mainly interested in the cases where $i = 2j-1$ and where $\mathbb{A} = \mathbb{Q}$ and $\mathbb{A} = \mathbb{R}$. In these cases we have

$$H^{2j-1}_{\mathcal{D}}(X,\mathbb{Q}(j)) \simeq \frac{H^{2j-2}(X,\mathbb{C})}{F^j H^{2j-2}(X,\mathbb{C}) + H^{2j-2}(X,\mathbb{Q}(j))},$$
$$H^{2j-1}_{\mathcal{D}}(X,\mathbb{R}(j)) \simeq H^{j-1,j-1}(X,\mathbb{R}(j-1)).$$

(2) *Higher Chow groups.* For $X$ given in (1), the following abridged definition of $\mathrm{CH}^k(Z,1)$ will suffice (see [La] or [MS]).

*Definition* (2.2). $\mathrm{CH}^k(X,1)$ is the homology of the middle term in the complex

$$\coprod_{\mathrm{cd}_X Y = k-2} K_2(\mathbb{C}(Y)) \xrightarrow{\text{Tame}} \coprod_{\mathrm{cd}_X Y = k-1} K_1(\mathbb{C}(Y)) \xrightarrow{\text{Div}} \coprod_{\mathrm{cd}_X Y = k} K_0(\mathbb{C}(Y)),$$

where we recall that $K_1(\mathbb{F}) = \mathbb{F}^\times$ and $K_0(\mathbb{F}) = \mathbb{Z}$, for a field $\mathbb{F}$, and Tame, Div are respectively the Tame symbol and divisor maps.

Thus classes in $\mathrm{CH}^k(X,1)$ can be represented by cycles of the form

$$\left\{ \xi = \sum_{j=1}^N \big(f_j, Z_j\big) \;\middle|\; f_j \in \mathbf{C}(Z_j)^\times,\ \mathrm{cd}_X = k-1,\ \sum_{j=1}^N \mathrm{div}_{Z_j}(f_j) = 0 \right\}.$$

*Note:* For the most part, we will identify $\mathrm{CH}^k(-,m)$ with $\mathrm{CH}^k(-,m)\otimes\mathbb{Q}$, unless there is a specific reason to work with $\mathrm{CH}^k(-,m)$ (and in which case the interpretation will be clear).

(3) *Regulators.* There are cycle class maps

$$\mathrm{cl}^{\mathbb{A}}_{k,1} \colon \mathrm{CH}^k(X,1) \to H^{2k-1}_{\mathcal{D}}(X,\mathbb{A}(k)).$$

In the case where $\mathbb{A} = \mathbb{R}$, we put $r_{k,1} = \mathrm{cl}^{\mathbb{R}}_{k,1}$. Let $n = \dim X$. The map

$$r_{k,1} \colon \mathrm{CH}^k(X,1) \to H^{2k-2}_{\mathcal{D}}(X,\mathbb{R}(k)) \simeq H^{k-1,k-1}(X,\mathbb{R}(k-1))$$
$$\simeq H^{n-k+1,n-k+1}(X,\mathbb{R}(n-k+1))^\vee,$$

is given explicitly as follows (see [Bei1] or [Ja]):

$$r_{k,1}(\xi)(\omega) = \frac{1}{(2\pi\sqrt{-1})^{n-k+1}} \sum_{j=1}^N \int_{Z_j} \omega \log|f_j|.$$

(4) *Horizontal displacement.* Let $h \colon W \to S$ be a proper smooth morphism of quasiprojective varieties over $\mathbb{C}$, where say for simplicity $\dim S = 1$, with smooth projective fiber $W_t := h^{-1}(t)$. Fix a reference point $t_0 \in S$ and consider a disk $\Delta$ centered at $t_0$. It is well known that there is a diffeomorphism $h^{-1}(\Delta) \approx \Delta \times W_{t_0}$. Thus for a cohomology class $\gamma := \gamma_{t_0} \in H^\bullet(W_{t_0})$, one can talk about its horizontal displacement $\gamma_t \in H^\bullet(W_t)$, for $t \in \Delta$ and more generally for $t \in S$. Consider the Hodge decomposition $H^\bullet(W_t,\mathbb{C}) = \bigoplus_{p+q=\bullet} H^{p,q}(W_t)$, $\gamma_t = \oplus_{p+q=\bullet}\gamma^{p,q}$. We say that the Hodge $(p,q)$ components deform horizontally

if $\gamma_t^{p,q} = (\gamma^{p,q})_t$ for all $t \in \Delta$. By analytic considerations of Hodge subbundles, this is equivalent to saying that $\gamma_t^{p,q} = (\gamma^{p,q})_t$ for all $t \in S$.

(5) The word "general" in this paper will have the following meaning. In the notation of (0.1), a point $t \in \mathcal{T}$ is general if $t$ belongs to the complement of a *countable* union of proper subvarieties of $\mathcal{T}$ governed by a certain property.

## 3. Proof of Theorem (1.1)

Let $\{X_t\}_{t\in\mathbb{P}^1}$ be a Lefschetz pencil of surfaces of degree $d \geq 5$ in $\mathbb{P}^3$, i.e. the general fiber $X_t$ is smooth, and each singular fiber has an ordinary double point singularity. We will think of this pencil in the form $X \subset \mathbb{P}^3 \times \mathbb{P}^1$, i.e. where $X$ is the blowup of $\mathbb{P}^3$ along the base locus $\cap_{t\in\mathbb{P}^1} X_t$. Suppose that for a general $t \in \mathbb{P}^1$, the cycle class map $\mathrm{cl}_{2,1}\colon \mathrm{CH}^2(X_t, 1) \to H^3_{\mathcal{D}}(X_t, \mathbb{Q}(2))$ is nontrivial. We can assume that $X$ is defined over an algebraically closed field $L$ of finite transcendence degree over $\mathbb{Q}$, i.e. $X/_{\mathbb{C}} = X_L \times \mathbb{C}$. Let $\eta$ be the generic point of $\mathbb{P}^1_L$. For some finite algebraic extension $K \supset L(\eta)$, and via a suitable embedding $K \hookrightarrow \mathbb{C}$, there is a class $\xi_K \in \mathrm{CH}^2(X_K := X_\eta \times K, 1)$ such that $\mathrm{cl}_{2,1}(\xi_K) \neq 0$ in $H^3_{\mathcal{D}}(X_K(\mathbb{C}), \mathbb{Q}(2))$. The situation here is not unlike than that found in ([Lew] p. 191). There is a smooth projective curve $\Gamma_L$ with function field $L(\Gamma) = K$. Then after a base change $Y = X \times_{\mathbb{P}^1} \Gamma$, $\xi_K$ defines a cycle in $\xi \in \mathrm{CH}^2(Y_U, 1)$, where $U \subset \Gamma$ is a Zariski open subset of $\Gamma$ and $Y_U = \cup_{t\in U} Y_t$. This uses the fact that

$$\mathrm{CH}^2(X_K, 1) = \mathrm{CH}^2(Y_{\tilde{\eta}}, 1) = \varinjlim_{U} \mathrm{CH}^2(Y_U, 1),$$

where $Y_{\tilde{\eta}}$ is the generic fiber of $Y$ over $\Gamma_L$. We want to spread $\xi$ to *all* of $\Gamma$. However, there is obstruction preventing us to do it; rather we can extend it after a suitable modification of $\xi$. That is, we will show that there exists $\xi' \in \mathrm{CH}^2(Y, 1)$ such that $\mathrm{cl}_{2,1}(\xi_t) = \mathrm{cl}_{2,1}(\xi'_t)$ for every $t \in U$. Our main tool is the localization sequence. (Strictly speaking, we don't really need the localization sequence in this paper. Rather, it is used out of convenience).

$$\mathrm{CH}^2(Y, 1) \to \mathrm{CH}^2(Y_U, 1) \to \mathrm{CH}^1(Y_B) \to \mathrm{CH}^2(Y) \to \mathrm{CH}^2(Y_U) \to 0 \tag{3.1}$$

over $\mathbb{Q}$, where $B = \Gamma \backslash U$ and $Y_B = \cup_{t\in B} Y_t$.

Note that the map $\mathrm{CH}^1(Y_B) \to \mathrm{CH}^2(Y)$ might not be injective if $|B| > 1$, so there is obstruction to extend $\xi$ directly.

Let $H$ be a plane in $\mathbb{P}^3$ and $\pi^*H \subset Y$ be the pullback of $H$ under the projection $\pi\colon Y \to \mathbb{P}^3$. Let $C_b = \pi^*H \cap Y_b$ for $b \in B$ and $C_B = \cup_{b\in B} C_b$. Let us first extend $\xi$ to $Y \backslash C_B$. We look at the localization sequence

$$\mathrm{CH}^2(Y\backslash C_B, 1) \to \mathrm{CH}^2(Y_U, 1) \to \mathrm{CH}^1(Y_B\backslash C_B) \to \mathrm{CH}^2(Y\backslash C_B) \tag{3.2}$$

Note that

$$\mathrm{CH}^1(Y_B\backslash C_B) = \bigoplus_{b\in B} \mathrm{CH}^1(Y_b\backslash C_b) \tag{3.3}$$

We claim that $\mathrm{CH}^1(Y_t\backslash C_t) \otimes \mathbb{Q} = 0$ for every $t \in \Gamma$.

The classical Noether-Lefschetz theorem tells us that a general surface of degree $d \geq 4$ in $\mathbb{P}^3$ has Picard rank 1. This statement was refined by Mark Green [G] to the following. Let $M = \mathbb{P}^N$ be the space parameterizing surfaces of degree $d$ in $\mathbb{P}^3$ and $M_2 \subset M$ be the subset parameterizing surfaces with Picard rank $\geq 2$. Then $\operatorname{codim}_M M_2 = d-3$. So when $d \geq 5$, $M_2$ has codimension at least 2 in $M$ and a general pencil will avoid this locus. Thus $\operatorname{Pic}(Y_t)\otimes\mathbb{Q} = \mathbb{Q}$ for every $t\in\Gamma$. Note that $Y_t$ might be singular, i.e., $Y_t$ has an ordinary double point. Since an ordinary double point is a quotient singularity, every Weil divisor of $Y_t$ is $\mathbb{Q}$-Cartier. Therefore, $\mathrm{CH}^1(Y_t)\otimes\mathbb{Q} = \operatorname{Pic}(Y_t)\otimes\mathbb{Q}$. In any case, we have

$$\mathrm{CH}^1(Y_t)\otimes\mathbb{Q} = \operatorname{Pic}(Y_t)\otimes\mathbb{Q} = \operatorname{Pic}(\mathbb{P}^3)\otimes\mathbb{Q} = \mathbb{Q}. \tag{3.4}$$

Obviously, $\mathrm{CH}^1(Y_t)$ is generated by $C_t = \pi^*H\cap Y_t$ over $\mathbb{Q}$. Consequently,

$$\mathrm{CH}^1(Y_t\backslash C_t)\otimes\mathbb{Q} = 0 \tag{3.5}$$

and there is no obstruction to extend $\xi$ to $Y\backslash C_B$. So we may regard $\xi$ as a class in $\mathrm{CH}^2(Y\backslash C_B, 1)$ from now on.

There might be obstruction to further extend $\xi$ to all of $Y$ by the localization sequence

$$\mathrm{CH}^2(Y,1)\to \mathrm{CH}^2(Y\backslash C_B,1)\xrightarrow{\phi}\mathrm{CH}^0(C_B)\xrightarrow{\gamma}\mathrm{CH}^2(Y) \tag{3.6}$$

where

$$\mathrm{CH}^0(C_B) = \bigoplus_{b\in B}\mathrm{CH}^0(C_b) = \mathbb{Q}^{\oplus\beta} \tag{3.7}$$

with $\beta = |B|$.

Let $\xi = \sum_\alpha (f_\alpha, D_\alpha)$ where $D_\alpha$ is a divisor on $Y\backslash C_B$ and $f_\alpha$ is a rational function on $D_\alpha$. We have

$$\sum_\alpha \operatorname{div}(f_\alpha) = 0. \tag{3.8}$$

Let $\overline{D}_\alpha$ be the closure of $D_\alpha$ in $Y$ and $f_\alpha$ naturally extends to a rational function $\overline{f}_\alpha$ on $\overline{D}_\alpha$. Let $\overline{\xi} = \sum_\alpha(\overline{f}_\alpha, \overline{D}_\alpha)$. We no longer have (3.8). Instead,

$$\sum_\alpha \operatorname{div}(\overline{f}_\alpha) = \sum_{b\in B} m_b C_b \tag{3.9}$$

for some $m_b\in\mathbb{Z}$. Actually, the RHS of (3.9) is exactly the image of $\xi$ under the map $\phi$: $\mathrm{CH}^2(Y\backslash C_B,1)\to\mathrm{CH}^0(C_B)$ in (3.6), i.e.,

$$\phi(\xi) = \sum_{b\in B} m_b C_b. \tag{3.10}$$

Note that $\phi(\xi)$ lies in the kernel of $\gamma$: $\mathrm{CH}^0(C_B)\to\mathrm{CH}^2(Y)$ and there is a natural map $\mathrm{CH}^0(C_B)\to\mathrm{CH}^1(\Gamma)$ via

$$\mathrm{CH}^0(C_B)\xrightarrow{\gamma}\mathrm{CH}^2(Y)\to\mathrm{CH}^3(\mathbb{P}^3\times\Gamma)\to\mathrm{CH}^1(\Gamma). \tag{3.11}$$

Note that the map $\mathrm{CH}^3(\mathbf{P}^3\times\Gamma)\to[\mathbb{P}^1]\otimes\mathrm{CH}^1(\Gamma) = \mathrm{CH}^1(\Gamma)$, comes from the projective bundle formula. Of course, the map $\mathrm{CH}^0(C_B)\to\mathrm{CH}^1(\Gamma)$ simply sends $C_b$ to $Nb$, where $N = d$. Thus $\phi(\xi)$ maps to zero under this map, i.e. the divisor $\sum m_b b$ is $N$-torsion in $\mathrm{CH}^1(\Gamma) = \operatorname{Pic}(\Gamma)$.

Note that $\pi^*H$ is a fibration of curves over $\Gamma$. So the fact $\sum m_b b$ is torsion in $\mathrm{CH}^1(\Gamma)$ implies that $\sum m_b C_b$ is $N$-torsion in $\mathrm{CH}^1(\pi^*H)$. Consequently, there exists a rational function $f_H$ on $\pi^*H$ such that

$$\operatorname{div}(f_H) = N\sum_{b\in B} m_b C_b. \tag{3.12}$$

So we may simply modify $\overline{\xi}$ as follows

$$\xi' = \overline{\xi} - \frac{1}{N}(f_H, \pi^* H). \tag{3.13}$$

Now $\xi' \in \mathrm{CH}^2(Y,1)$ and $\underline{\mathrm{cl}}_{2,1}(\xi'_t) = \underline{\mathrm{cl}}_{2,1}(\xi_t)$ for all $t \in U$, where we recall that

$$\underline{\mathrm{cl}}_{2,1}\colon\ \mathrm{CH}^2(Y_t,1) \to \frac{H^3_{\mathcal{D}}(Y_t,\mathbb{Q}(2))}{\mathrm{Hg}^1(Y_t)\otimes\big(\mathbb{C}/\mathbb{Q}(1)\big)}$$

is the induced map. This is due to the fact that the restrictions $f_H$ to $Y_t$ are obviously constants. Thus we can now replace $\xi$ by $\xi'$. Next observe that even though $Y$ is complete, it may be singular. It is worthwhile pointing out that we can further pull back $\xi$ to a desingularization $\widetilde{Y}$ of $Y$. More precisely,

*Claim* (3.14). There exists $\widetilde{\xi} \in \mathrm{CH}^2(\widetilde{Y},1)$ such that $\widetilde{\xi}$ and $\xi$ agree on the open set where $\widetilde{Y}$ and $Y$ are isomorphic.

The usefulness of this claim is as follows. The (cohomological) cycle class map $\mathrm{cl}_{2,1}\colon\ \mathrm{CH}^2(Y,1) \to H^3_{\mathcal{D}}(Y,\mathbb{Q}(2))$ is only defined if $Y$ is smooth. Granting the existence of this cycle class map, the remaining argument only requires the completeness of $Y$. There is a short exact sequence:

$$0 \to \frac{H^2(Y,\mathbb{C})}{F^2H^2(Y,\mathbb{C}) + H^2(Y,\mathbb{Q}(2))} \to H^3_{\mathcal{D}}(Y,\mathbb{Q}(2)) \to F^2 \cap H^3(Y,\mathbb{Q}(2)) \to 0.$$

But since $Y$ is complete, a weight argument gives $F^2 \cap H^3(Y,\mathbb{Q}(2)) = 0$. Thus for $t \in U$, $\underline{\mathrm{cl}}_{2,1}(\xi_t)$ is given by the restriction $\underline{\mathrm{cl}}_{2,1}(\xi)\big|_{Y_t}$, i.e. induced by the restriction

$$\frac{H^2(Y,\mathbb{C})}{F^2H^2(Y,\mathbb{C}) + H^2(Y,\mathbb{Q}(2))} \to \frac{H^2(Y_t,\mathbb{C})}{F^2H^2(Y_t,\mathbb{C}) + H^2(Y_t,\mathbb{Q}(2))}.$$

Thus as $t \in U$ varies, the class $\underline{\mathrm{cl}}_{2,1}(\xi_t)$ varies by *horizontal* displacement; further, the restriction $H^2(Y) \to H^2(Y_t)$ is a morphism of mixed Hodge structures. Thus $\underline{\mathrm{cl}}_{2,1}(\xi_t)$ is induced by a class in $H^2(Y_t)$, whose Hodge $(p,q)$ components displace *horizontally*, i.e. preserving the given Hodge type. But over the set where $\Gamma \to \mathbb{P}^1$ ramifies, one can find open sets $\Delta_\Gamma \subset U \subset \Gamma$, $\Delta \subset \mathbb{P}^1$, in the strong topology, such that $\Delta_\Gamma \simeq \Delta$. Thus $\underline{\mathrm{cl}}_{2,1}(\xi_t) = 0$, by virtue of:

LEMMA (3.15). *Consider a Lefschetz pencil $\{Z_t\}_{t\in\mathbb{P}^1}$ of surfaces in $\mathbb{P}^3$ of degree $d \geq 1$, and let $U_0 \subset \mathbb{P}^1$ be the smooth set. Further, let $\Delta \subset U_0$ be a disk, and assume given $\gamma_t \in H^2(Z_t,\mathbb{C})$, a horizontal displacement of a class $\gamma$ for $t \in \Delta$. If the $(p,q)$ components of $\gamma_t$ also horizontally displace, then $\gamma_t \in \mathrm{Hg}^1(Z_t)$.*

*Proof.* This follows from a standard monodromy argument, together with the analyticity of Hodge subbundles. □

Finally, we attend to:

*Proof of claim.* It turns out that the singularities of $Y$ are quite mild. Note that the singularities of $Y$ are introduced during the base change $\Gamma \to \mathbb{P}^1$; $Y$ becomes singular when the map $\Gamma \to \mathbb{P}^1$ ramifies over a point $t \in \mathbb{P}^1$ where $X_t$ is singular, i.e., it has an ordinary double point. Therefore, the singularities of $Y$ have the type of $x^2 + y^2 + z^2 + t^m = 0$. Let $p \in Y$ be such a singularity. We may solve $p$ by a sequence of blowups:

$$\widetilde{Y} = Y_\mu \xrightarrow{\varphi_\mu} Y_{\mu-1} \xrightarrow{\varphi_{\mu-1}} ... \xrightarrow{\varphi_1} Y_0 = Y \tag{3.16}$$

where $\mu = \lfloor m/2 \rfloor$. The exceptional divisor $E_k \subset Y_k$ of $\varphi_k$ is a quadric in $\mathbb{P}^3$; it is a cone over a conic curve if $2k < m$ and it is a smooth quadric if $m = 2k$. Let $p_0 = p$ and $p_k \in E_k$ be the vertex of the cone $E_k$ for $2k < m$. It is obvious that $Y_k$ is locally given by $x^2 + y^2 + z^2 + t^{m-2k} = 0$ at $p_k$ and $\varphi_{k+1}\colon Y_{k+1} \to Y_k$ is the blowup of $Y_k$ at $p_k$.

In order to pull back $\xi$ to $\widetilde{Y}$, we do it step by step, i.e., we first pull it back to $Y_1$, then $Y_2$ and so on. We will show that there exists a sequence of cycles $\{\xi_k \in \mathrm{CH}^2(Y_k, 1)\}$ with all of them agreeing on the open set $Y\backslash\{p\}$.

By induction, it suffices to pull back the cycle $\xi_{k-1} \in \mathrm{CH}^2(Y_{k-1}, 1)$ to $\xi_k \in \mathrm{CH}^2(Y_k, 1)$.

Since $\varphi_k\colon Y_k \to Y_{k-1}$ is the blowup of $Y_{k-1}$ at $p_{k-1}$,

$$Y_k\backslash E_k \cong Y_{k-1}\backslash\{p_{k-1}\}. \tag{3.17}$$

So the question is again to extend a class in $\mathrm{CH}^2(Y_k\backslash E_k, 1)$ to $\mathrm{CH}^2(Y_k, 1)$. We look at the localization sequence

$$\mathrm{CH}^2(Y_k, 1) \to \mathrm{CH}^2(Y_k\backslash E_k, 1) \to \mathrm{CH}^1(E_k) \xrightarrow{\gamma} \mathrm{CH}^2(Y_k)\,. \tag{3.18}$$

If $E_k$ is a cone over a conic curve, then $\mathrm{CH}^1(E_k) = \mathbb{Q}$ (see [Ha, Appendix A, Example 1.1.2, p. 428]) and $\gamma\colon\ \mathrm{CH}^1(E_k) \to \mathrm{CH}^2(Y_k)$ is obviously injective.

Suppose that $E_k$ is a smooth quadric. This happens in the last step of blowups, i.e., when $k = \mu$ and $m = 2\mu$ is even. Now

$$\mathrm{CH}^1(E_k) = \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Q} \oplus \mathbb{Q}. \tag{3.19}$$

Let $L_1, L_2 \subset E_k$ be the two rulings of $E_k$ which generate $\mathrm{CH}^1(E_k)$. We claim that $L_1$ and $L_2$ are numerically independent on $Y_k$, i.e., there exist divisors $D_1, D_2 \subset Y_k$ such that $D_i \cdot L_j = 0$ if $i = j$ and $D_i \cdot L_j \neq 0$ if $i \neq j$. This certainly implies that $\gamma$ is injective.

Note that $Y_{k-1}$ has an ordinary double point $x^2 + y^2 + z^2 + t^2 = 0$ at $p_{k-1}$. It is well known that there exist two small resolutions of $Y_{k-1}$. That is, we may blow down $Y_k$ along either of the two rulings $L_1$ and $L_2$. Let $g\colon Y_k \to Y_k'$ be the blowdown of $Y_k$ along $L_1$. Let $D$ be an ample divisor on $Y_k'$. Then $g^*D \cdot L_2 \neq 0$ since $D$ is ample on $Y_k'$ and $g^*D \cdot L_1 = 0$ since $g_*L_1 = 0$. We are done. ☐

## 4. Proof of Theorem (1.2)

We see that the very essence of the proof for Theorem (1.1), i.e., the fact that the argument works for quintic surfaces but not quartic surfaces, lies in the result of M. Green that those quintic surfaces with Picard rank two lie in a subset of codimension two in the moduli space of quintic surfaces, while the same is not true for quartic surfaces. For a product of curves $C_1 \times C_2$ with $g(C_1)g(C_2) > 1$, we have a similar situation. It is well known that $\mathrm{Pic}(C_1 \times C_2) = \mathrm{Pic}(C_1) \oplus \mathrm{Pic}(C_2)$ for a general pair $(C_1, C_2)$. Moreover, we have the following

Proposition (4.1). *Let* $g_1 = g(C_1)$ *and* $g_2 = g(C_2)$ *with* $g_1 g_2 > 1$ *and* $\mathcal{M}_{g_i}$ *be the moduli space of curves of genus* $g_i$*. Let* $W \subset \mathcal{M}_{g_1} \times \mathcal{M}_{g_2}$ *be the locus of the products* $C_1 \times C_2$ *with* $\rho(C_1 \times C_2) > 2$*, where* $\rho(S)$ *is the rank of the Neron-Severi group of* $S$*. Then* $\operatorname{codim} W \geq 2$.

We believe that the above proposition is well known. But since we cannot locate a reference to it, we will give a proof at the end of this section.

Fix $E = C_2$ be and $Y \to \Gamma$ be a one parameter family of curves of genus $g_1 = g(C_1)$. For a general choice of $Y$, we assume that

$$\mathrm{CH}^1(Y_t \times E) = \mathrm{CH}^1(Y_t) \oplus \mathrm{CH}^1(E) \tag{4.2}$$

for every $t \in \Gamma$ by Proposition (4.1), where $Y_t$ is the fiber over a point $t \in \Gamma$. Here we have to be a little careful about the singular fibers of $Y$ as Proposition (4.1) does not say anything about one of $C_i$ being singular. Let $C = Y_s$ be a singular fiber of $Y$. For a general choice of $Y$, $C$ has one node and its normalization $\widetilde{C}$ is a general curve of genus $g_1 - 1$. Therefore, $\mathrm{CH}^1(\widetilde{C} \times E) = \mathrm{CH}^1(\widetilde{C}) \oplus \mathrm{CH}^1(E)$ and $\mathrm{CH}^1(C \times E) \cong \mathrm{CH}^1(\widetilde{C} \times E)/\{F_p \sim F_q\}$. where $F_p$ and $F_q$ are the fibers of $\widetilde{C} \times E$ over $p, q \in \widetilde{C}$ which are the two points over the node of $C$. Hence (4.2) follows for singular fibers $Y_s$. This is still true after a base change of $Y$ followed by a semi-stable reduction, in which case $Y_s$ is a union of curves $R_0 \cup R_1 \cup \ldots \cup R_n$ with $g(R_0) = g_1 - 1$, $g(R_1) = \ldots = g(R_n) = 0$ and $R_i R_{i+1} = R_0 R_n = 1$.

For a subset $U \subset \Gamma$, we use the notation $Y_U$ for $Y_U = \cup_{t \in U} Y_t$.

Let $\xi \in \mathrm{CH}^2(Y_U \times E, 1)$ for some open set $U \subset \Gamma$. We claim that $\xi$ can be extended to everywhere on $Y$, i.e., there exists $\overline{\xi} \in \mathrm{CH}^2(Y \times E, 1)$ such that $\underline{\mathrm{cl}}_{2,1}(\xi_t) = \underline{\mathrm{cl}}_{2,1}(\overline{\xi}_t)$ for a general point $t \in \Gamma$, where $\xi_t$ and $\overline{\xi}_t$ are the restrictions $\xi$ and $\widetilde{\xi}$ to the fiber $Y_t \times E$, respectively.

Let $B = \Gamma \backslash U$. We have the localization sequence

$$\mathrm{CH}^2(Y \times E, 1) \to \mathrm{CH}^2(Y_U \times E, 1) \xrightarrow{\phi} \mathrm{CH}^1(Y_B \times E) \xrightarrow{\gamma} \mathrm{CH}^2(Y \times E)\,. \tag{4.3}$$

Let $\xi = \sum_\alpha (D_\alpha, f_\alpha)$, where $D_\alpha$ is a divisor on $Y_U \times E$ and $f_\alpha$ is a rational function on $D_\alpha$. We have

$$\mathrm{div}(\xi) = \sum_\alpha \mathrm{div}(f_\alpha) = 0. \tag{4.4}$$

Let $\overline{D}_\alpha$ be the closure of $D_\alpha$ in $Y \times E$ and $f_\alpha$ naturally extends to a rational function $\overline{f}_\alpha$ on $\overline{D}_\alpha$. Let $\overline{\xi} = \sum_\alpha (\overline{D}_\alpha, \overline{f}_\alpha)$. We no longer have $\mathrm{div}(\overline{\xi}) = 0$. Instead,

$$\mathrm{div}(\overline{\xi}) = \sum_\alpha \mathrm{div}(\overline{f}_\alpha) \in Z^1(Y_B \times E) \tag{4.5}$$

where $Z^k(X)$ is the free abelian group generated by the codimension-$k$ algebraic cycles of $X$. Obviously, $\mathrm{div}(\overline{\xi})$ is exactly the image of $\phi$ in the localization sequence (4.3).

By our assumption,

$$\mathrm{CH}^1(Y_B \times E) = \left(\mathrm{CH}^1(Y_B) \otimes \mathrm{CH}^0(E)\right) \oplus \left(\mathrm{CH}^0(Y_B) \otimes \mathrm{CH}^1(E)\right). \tag{4.6}$$

Therefore,

$$\sum_\alpha \operatorname{div}(\overline{f}_\alpha) \sim_{rat} D_Y \otimes E + \sum_{b\in B} Y_b \otimes D_b \tag{4.7}$$

where $\sim_{rat}$ is the rational equivalence relation, $D_Y \in \mathrm{CH}^1(Y_B)$ and $D_b \in \mathrm{CH}^1(E)$. Therefore, there exist rational functions $g_b$ on $Y_b \times E$ such that

$$\sum_\alpha \operatorname{div}(\overline{f}_\alpha) + \sum_{b\in B} \operatorname{div}(g_b) = D_Y \otimes E + \sum_{b\in B} Y_b \otimes D_b \tag{4.8}$$

Hence we may replace $\overline{\xi}$ by $\overline{\xi} + \sum_{b\in B}(Y_b \times E, g_b)$ and assume that

$$\operatorname{div}(\overline{\xi}) = D_Y \otimes E + \sum_{b\in B} Y_b \otimes D_b\,. \tag{4.9}$$

Since $\gamma(\operatorname{div}(\overline{\xi})) = 0$,

$$D_Y \otimes E + \sum_{b\in B} Y_b \otimes D_b \sim_{rat} 0 \tag{4.10}$$

in $\mathrm{CH}^2(Y \times E)$. Now choose any point $p \in (E\backslash \cup_{b\in B} D_b)$. Then $D_Y \times p$ is the expression in (4.9) intersected with $Y \times p$. But the expression in (4.9) being $\sim_{rat}$ 0 implies that the intersection cycle is $\sim_{rat}$ 0, i.e. $\{D_Y \times p\} \sim_{rat} 0$. Thus $D_Y = Pr_{Y,*}(D_Y \times p) \sim_{rat} 0$ in $\mathrm{CH}^2(Y)$. Next, by definition of rational equivalence, if $D \in Z^2(Y)$ and $D \sim_{rat} 0$, then there exists a pre-higher Chow cycle $\varepsilon = \sum_\beta (D_\beta, f_\beta)$ on $Y$ with $\operatorname{div}(\varepsilon) = D$. Therefore, there exists $\varepsilon = \sum_\beta (D_\beta, f_\beta)$ on $Y$ with $\operatorname{div}(\varepsilon) = D_Y$. So we may replace $\overline{\xi}$ by

$$\overline{\xi} - \varepsilon \otimes E = \overline{\xi} - \sum_\beta (D_\beta \times E, f_\beta \times E) \tag{4.11}$$

with resulting $\overline{\xi}$ satisfying

$$\operatorname{div}(\overline{\xi}) = \sum_{b\in B} Y_b \otimes D_b\,. \tag{4.12}$$

Obviously,

$$\sum_{b\in B} Y_b \otimes D_b = \pi^*(\delta) \tag{4.13}$$

for some $\delta \in Z^2(\Gamma \times E)$, where $\pi\colon Y \times E \to \Gamma \times E$ is the projection induced by $Y \to \Gamma$.

Note that if $\pi\colon X \to Y$ be a surjective morphism between two smooth projective varieties. Then

$$\mathrm{CH}^k(Y) \otimes \mathbb{Q} \xrightarrow{\pi^*} \mathrm{CH}^k(X) \otimes \mathbb{Q} \tag{4.14}$$

is injective. This follows by reducing to the case where $\dim Y = \dim X$, and using the fact that $\pi_* \circ \pi^* = \deg \pi$.

Thus there exists a pre-higher Chow cycle $\varepsilon$ on $\Gamma \times E$ with $\operatorname{div}(\varepsilon) = N\delta$. Finally, we replace $\overline{\xi}$ by

$$\overline{\xi} - \frac{1}{N}\pi^*\varepsilon \tag{4.15}$$

and obtain a higher Chow cycle $\overline{\xi} \in \mathrm{CH}^2(Y \times E, 1)$. It is easy to check that $\underline{\mathrm{cl}}_{2,1}(\xi_t) = \underline{\mathrm{cl}}_{2,1}(\overline{\xi}_t)$ for a general $t \in \Gamma$.

Next we use a monodromy argument just like in the proof of Theorem (1.1). Applying the same monodromy action considerations to $\pi(\{s \in \Gamma \colon Y_s \text{ is smooth}\}) \to \mathrm{Aut}\big(H^1(Y_t)\big)$, any class in $H^2(Y_t \times E)$ whose Hodge $(p,q)$ components deform horizontally, must be algebraic. But since $\underline{\mathrm{cl}}_{2,1}(\overline{\xi}_t)$ is induced by restriction from a class $\underline{\mathrm{cl}}_{2,1}(\overline{\xi})$, and hence from a cohomology class in $H^2(Y \times E)$, it is clear that the Hodge $(p,q)$ components of $\underline{\mathrm{cl}}_{2,1}(\overline{\xi}_t)$ deform horizontally as the restriction $H^2(Y \times E) \to H^2(Y_t \times E)$ is a morphism of Hodge structures, *a fortiori* $\underline{\mathrm{cl}}_{2,1}(\overline{\xi}_t) = 0$ for general $t$. We are done.

It remains to give a proof for Proposition (4.1). We will use a deformation-theoretic argument.

Lemma (4.16). *Let $X/\Delta$ be a family of smooth projective surfaces over disk $\Delta$ with central fiber $S = X_0$ and let $D \subset S$ be an effective divisor on $S$. Suppose that $D$ can be extended to $X$, i.e., there exists a flat family $Y/\Delta$ with the commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ \Delta \arrow[r] & \Delta \end{tikzcd} \tag{4.17}$$

*such that $Y_0$ embeds into $X_0$ with image $D$. For each $w \in H^0(K_S)$, let $\mu_w$ be the map*

$$\mu_w \colon H^1(\Omega_S) \xrightarrow{\otimes w} H^1(\Omega_S(K_S)) \tag{4.18}$$

*where $K_S$ is the canonical class of $S$. Then the Kodaira-Spencer class* $\mathrm{ks}(\partial/\partial t) \in H^1(T_S)$ *of $X$ lies in the subspace*

$$\{v \in H^1(T_S) \colon \langle v, \mu_w(c_1(D))\rangle = 0 \textit{ for all } w \in H^0(K_S)\} \tag{4.19}$$

*where $\langle \cdot, \cdot \rangle$ is the pairing $H^1(T_S) \times H^1(\Omega_S(K_S)) \to \mathbb{C}$ given by Serre duality.*

*Proof.* The pushforward $\pi_* Y$ is a divisor on $X$ whose restriction to $X_0$ is a multiple of $D$, say $nD$. Fix a sufficiently ample divisor $A$ of $X$ and we embed $X$ to $\mathbb{P}^g \times \Delta$ with the linear series $|A + \pi_* Y|$. Let $N_S$ be the normal bundle of $S$ in $\mathbb{P}^g$. Then the Kodaira-Spencer map $\mathrm{ks} \colon T_{\Delta,0} \to H^1(T_S)$ factors through $H^0(N_S)$. Note the exact sequence

$$H^0(N_S) \to H^1(T_S) \xrightarrow{f} H^1(T_{\mathbb{P}^g}|_S). \tag{4.20}$$

We claim that the kernel of $f$ is contained in the space

$$\ker(f) \subset \{v \in H^1(T_S) \colon \langle v, \mu_w(c_1(A_0 + nD))\rangle = 0\} \tag{4.21}$$

where $A_0 + nD$ is the restriction of $A + \pi_* Y$ to $S$. Then $\mathrm{ks}(\partial/\partial t)$ lies in the space (4.21). The same argument with $A + \pi_* Y$ replaced by $A$ will produce that $\mathrm{ks}(\partial/\partial t)$ lies in

$$\{v \in H^1(T_S) \colon \langle v, \mu_w(c_1(A_0))\rangle = 0\}. \tag{4.22}$$

Then it follows that $\mathrm{ks}(\partial/\partial t)$ lies in the space (4.19). So it suffices to justify (4.21).

Consider the dual map

$$f^\vee\colon H^1(\Omega_{\mathbb{P}^g}(K_S)|_S) \to H^1(\Omega_S(K_S))\,. \tag{4.23}$$

Obviously, (4.21) is equivalent to the statement that the image of $f^\vee$ contains $\mu_w(c_1(A_0+nD))$.

From the Euler sequence, we see that $H^1(\Omega_{\mathbb{P}^g}|_S) = H^1(\Omega_{\mathbb{P}^g})$ is generated by $c_1(H)$, where $H$ is the hyperplane divisor of $\mathbb{P}^g$. Of course, $c_1(H)|_S = c_1(A_0 + nD)$. Hence the image of $H^1(\Omega_{\mathbb{P}^g}|_S) \to H^1(\Omega_S)$ is generated by $c_1(A_0+nD)$. From the commutative diagram

$$\begin{tikzcd} H^1(\Omega_{\mathbb{P}^g}|_S) \arrow[r, "\otimes w"] \arrow[d] & H^1(\Omega_{\mathbb{P}^g}(K_S)|_S) \arrow[d, "f^\vee"] \\ H^1(\Omega_S) \arrow[r, "\otimes w"] & H^1(\Omega_S(K_S)) \end{tikzcd} \tag{4.24}$$

we see that the image of $f^\vee$ contains $\mu_w(c_1(A_0+nD))$ for all $w \in H^0(K_S)$. $\square$

Now let us finish the proof of Proposition (4.1). First, let us deal with the case that $g_1, g_2 > 1$.

Let $D \subset S = C_1 \times C_2$ be an effective divisor with $D \notin \mathrm{CH}^1(C_1) \oplus \mathrm{CH}^1(C_2)$. Then under the decomposition

$$\begin{aligned} H^1(\Omega_S) = H^1(K_{C_1}) \otimes H^0(\mathcal{O}_{C_2}) \oplus H^0(K_{C_1}) \otimes H^1(\mathcal{O}_{C_2}) \\ \oplus H^0(\mathcal{O}_{C_1}) \otimes H^1(K_{C_2}) \oplus H^1(\mathcal{O}_{C_1}) \otimes H^0(K_{C_2}) \end{aligned} \tag{4.25}$$

the projection $\omega_1 + \omega_2$ of $c_1(D)$ to

$$H^0(K_{C_1}) \otimes H^1(\mathcal{O}_{C_2}) \oplus H^1(\mathcal{O}_{C_1}) \otimes H^0(K_{C_2}) \tag{4.26}$$

does not vanish, where

$$\omega_1 \in H^0(K_{C_1}) \otimes H^1(\mathcal{O}_{C_2}) \text{ and } \omega_2 \in H^1(\mathcal{O}_{C_1}) \otimes H^0(K_{C_2}). \tag{4.27}$$

The restriction of $\mu_w$ to the subspace (4.26) is

$$\mu_w\colon H^0(K_{C_1}) \otimes H^1(\mathcal{O}_{C_2}) \to H^0(2K_{C_1}) \otimes H^1(K_{C_2}) \tag{4.28}$$

and

$$\mu_w\colon H^1(\mathcal{O}_{C_1}) \otimes H^0(K_{C_2}) \to H^1(K_{C_1}) \otimes H^0(2K_{C_2}). \tag{4.29}$$

It is easy to see that the space spanned by $\mu_w(\omega_i)$ for $w \in H^0(K_S)$ has dimension $g_i$ if $\omega_i \neq 0$. Therefore, the space spanned by $\mu_w(c_1(D))$ has dimension at least $\min(g_1, g_2)$. By Lemma (4.16), the deformations of $S$ preserving $D$ have codimension at least $\min(g_1, g_2)$ in the versal deformation space of $S$ (the deformations of $S$ are unobstructed when $g_1, g_2 > 1$).

Suppose that one of $C_1$ and $C_2$ is elliptic. Let $E = C_2$ be elliptic and let $B = C_1$. Let $L$ be a line bundle on $S = B \times E$. For each $b \in B$, let $S_b$ be the fiber of $S$ over $b \in B$ and $L_b$ be the restriction $L$ to $S_b \cong E$. This gives a map $B \to \mathrm{Pic}(E)$ by sending $b \to L_b$. By fixing a base point on $E$, we obtain a map $\phi\colon B \to J(E) = E$. If $\phi$ is constant, then it is easy to see that $L \in \mathrm{Pic}(B) \oplus \mathrm{Pic}(E)$ and we are done. If not, we have a nontrivial map from $B$ to $E$. Fix $E$ and we see that the locus of the curves of genus $g_1$ that dominates $E$ has dimension $2g_1 - 3$ ($\phi$ has $2g_1 - 2$ ramification points; $-1$ for the automorphism of $E$). Proposition (4.1) now follows.

*Received May 26, 2003*

*Final version received January 12, 2004*

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