

The Real Regulator for a Product of $K3$ Surfaces

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ABSTRACT. Let X be a general complex algebraic $K3$ surface, and $\mathrm{CH}^2(X, 1)$ Bloch's higher Chow group ([Blo1]). In [C-L2] it is proven that the real regulator $r_{2,1} : \mathrm{CH}^2(X, 1) \otimes \mathbb{R} \rightarrow H^{1,1}(X, \mathbb{R})$ is surjective (Hodge- \mathcal{D} -conjecture). The question addressed in this paper is whether a similar story holds for products of $K3$ surfaces. We prove a general regulator result for a product of two surfaces, and then deduce, under the assumptions of [a variant of] the Bloch-Beilinson conjecture regarding the injectivity of the Abel-Jacobi map for smooth quasiprojective varieties defined over number fields, that the (induced) regulator map $r_{3,1} : \mathrm{CH}^3(X \times Y, 1) \rightarrow \{H_{\mathrm{tr}}^2(X, \mathbb{R}) \otimes H_{\mathrm{tr}}^2(Y, \mathbb{R})\} \cap H^{2,2}(X \times Y)$ is trivial for a general pair of $K3$ surfaces, where $H_{\mathrm{tr}}^2(X, \mathbb{R})$ is the space of transcendental classes in $H^2(X, \mathbb{R})$.

0. Introduction

Let $f : W \rightarrow \Gamma$ be a dominating morphism of smooth complex projective varieties, with $\dim W = 3$ and $\dim \Gamma = 1$. For $t \in \Gamma$, put $W_t = f^{-1}(t)$, and $U = \{t \in \Gamma \mid W_t \text{ is smooth}\}$. Let $\Sigma = \Gamma \setminus U$. For $t \in \Sigma$ we will assume that the singular set of W_t is a finite set of ordinary double points. With respect to a polarization of W , and for $t \in U$, let $\mathrm{Prim}^2(W_t)$ be the primitive cohomology ([G-H], p. 122). For fixed $t_0 \in U$, we will assume that the monodromy group representation $\pi_1(U) \rightarrow \mathrm{Aut}(\mathrm{Prim}^2(W_{t_0}))$ is irreducible. Let us further assume that $\mathrm{Prim}^{2,0}(W_t) \neq 0$ for $t \in U$. Note that for general $t \in \Gamma$, $\mathrm{Prim}^2(W_t) = H_{\mathrm{tr}}^2(W_t)$, where the latter is the space of transcendental cycles (= orthogonal complement of the algebraic classes under the cup product). We prove the following:

THEOREM 0.1. *Given the above assumptions, let Y be a smooth surface with the property that $\mathrm{CH}^1(Y; \mathbb{Q}) \simeq \mathbb{Q}$. If*

$$(0.2) \quad \mathrm{CH}^2(W_t \times Y; \mathbb{Q}) = \sum_{i=0}^2 \mathrm{CH}^i(W_t; \mathbb{Q}) \otimes \mathrm{CH}^{2-i}(Y; \mathbb{Q}),$$

for every $t \in \Gamma$, then the (induced) regulator map

$$r_{3,1} : \mathrm{CH}^3(W_t \times Y, 1) \rightarrow \{\mathrm{Prim}^2(W_t, \mathbb{R}) \otimes H_{\mathrm{tr}}^2(Y, \mathbb{R})\} \cap H^{2,2}(W_t \times Y),$$

is zero for general $t \in \Gamma$.

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Next, we prove the following:

PROPOSITION 0.3. *Suppose that the complex Abel-Jacobi map is injective for smooth quasiprojective varieties defined over a number field (see Conjecture 3.0 below). If $\{W_t\}_{t \in \Gamma}$ is a general choice of 1-parameter K3 surfaces, and if Y is likewise a general choice of K3 surface, then statement (0.2) in Theorem 0.1 holds.*

Combining Proposition 0.3 and Theorem 0.1, we arrive at:

COROLLARY 0.4. *With the assumption in Proposition 0.3, let X and Y be a general pair of K3 surfaces. Then the (induced) regulator map*

$$r_{3,1} : \mathrm{CH}^3(X \times Y, 1) \rightarrow \{H_{\mathrm{tr}}^2(X, \mathbb{R}) \otimes H_{\mathrm{tr}}^2(Y, \mathbb{R})\} \cap H^{2,2}(X \times Y),$$

is trivial.

We leave it to the reader to generalize Proposition 0.3 and Corollary 0.4 to other classes of surfaces (for example general complete intersection surfaces). In light of our knowledge of the Hodge- \mathcal{D} -conjecture for general K3 surfaces ([**C-L2**]), the authors regard Corollary 0.4 as rather surprising.

1. Some Definitions

Unless otherwise stated, all varieties in this paper are complex varieties.

1.0 Higher Chow groups. The seminal definition of higher Chow groups $\mathrm{CH}^k(X, m)$ can be found in [**Blo1**]. We put $\mathrm{CH}^k(X, m; \mathbb{Q}) = \mathrm{CH}^k(X, m) \otimes \mathbb{Q}$. Note that $\mathrm{CH}^k(X, 0) = \mathrm{CH}^k(X)$, where the latter is the standard definition given in [**F**]. We are mainly interested in the case $m = 1$. For a quasiprojective Z , the following abridged definition of $\mathrm{CH}^k(Z, 1)$ will suffice [**La**] (cf. [**MS2**]).

DEFINITION 1.1. $\mathrm{CH}^k(Z, 1)$ is the homology of the middle term in the complex

$$\coprod_{\mathrm{cd}_Z Y = k-2} K_2(\mathbb{C}(Y)) \xrightarrow{\mathrm{Tame}} \coprod_{\mathrm{cd}_Z Y = k-1} K_1(\mathbb{C}(Y)) \xrightarrow{\mathrm{div}} \coprod_{\mathrm{cd}_Z Y = k} K_0(\mathbb{C}(Y)),$$

where we recall that $K_1(\mathbb{F}) = \mathbb{F}^\times$ and $K_0(\mathbb{F}) = \mathbb{Z}$, for a field \mathbb{F} , and Tame, div are respectively the tame symbol and divisor maps.

1.2 Real regulator. For a projective algebraic manifold X of dimension n , the following abridged definition of the real regulator

$$r_{k,1} : \mathrm{CH}^k(X, 1) \rightarrow H^{k-1, k-1}(X, \mathbb{R}) := H^{2k-2}(X, \mathbb{R}) \cap H^{k-1, k-1}(X),$$

will suffice. First of all, Serre duality gives

$$H^{k-1, k-1}(X, \mathbb{R}) \simeq H^{n-k+1, n-k+1}(X, \mathbb{R})^\vee.$$

Let $\xi = \sum_\alpha (f_\alpha, Z_\alpha)$ represent a class in $\mathrm{CH}^k(X, 1)$. Thus $\mathrm{codim}_X Z_\alpha = k-1$ and $\sum_\alpha \mathrm{div}_{Z_\alpha}(f_\alpha) = 0$ in X . For $\omega \in H^{n-k+1, n-k+1}(X, \mathbb{R})$ we define

$$(1.3) \quad r_{k,1}(\xi)(\omega) = \sum_\alpha \int_{Z_\alpha} \log |f_\alpha| \omega.$$

Up to a multiplicative factor of a Tate twist, this definition agrees with that given in [**Be**] and [**Ja1**].

Next, for a surface X , the subspaces $\text{Prim}^2(X, \mathbb{R})$, $H_{\text{tr}}^2(X, \mathbb{R})$ of $H^2(X, \mathbb{R})$ are self-dual under the cup product pairing on $H^2(X, \mathbb{R})$. Thus for smooth projective surfaces X and Y , there are induced regulator maps:

$$\begin{aligned} r_{3,1} : \text{CH}^3(X \times Y, 1) &\rightarrow H^{2,2}(X, \mathbb{R}) \\ &\rightarrow \left[\text{Prim}^2(X, \mathbb{R}) \otimes H_{\text{tr}}^2(Y, \mathbb{R}) \right] \cap H^{2,2}(X \times Y). \\ &\quad \left[H_{\text{tr}}^2(X, \mathbb{R}) \otimes H_{\text{tr}}^2(Y, \mathbb{R}) \right] \end{aligned}$$

1.4 Horizontal displacement. Let $h : W \rightarrow S$ be a proper smooth morphism of quasiprojective varieties over \mathbb{C} , where say for simplicity $\dim S = 1$, with smooth projective fiber $W_t := h^{-1}(t)$. Fix a reference point $t_0 \in S$ and consider a disk Δ centered at t_0 . It is well known that there is a diffeomorphism $h^{-1}(\Delta) \approx \Delta \times W_{t_0}$. For $t \in \Delta$, let $j_t : W_t \hookrightarrow h^{-1}(\Delta)$ be the inclusion. Then $j_t^* : H^\bullet(h^{-1}(\Delta), \mathbb{C}) \xrightarrow{\sim} H^\bullet(W_t, \mathbb{C})$ is an isomorphism. Thus for a cohomology class $\gamma := \gamma_{t_0} \in H^\bullet(W_{t_0})$, one can talk about its horizontal displacement $\gamma_t := j_t^* \circ (j_{t_0}^*)^{-1}(\gamma) \in H^\bullet(W_t)$, for $t \in \Delta$ and more generally for $t \in S$. Consider the Hodge decomposition $H^\bullet(W_t, \mathbb{C}) = \bigoplus_{p+q=\bullet} H^{p,q}(W_t)$, $\gamma_t = \bigoplus_{p+q=\bullet} \gamma_t^{p,q}$. We say that the Hodge (p, q) components deform horizontally if $\gamma_t^{p,q} = (\gamma^{p,q})_t$ for all $t \in \Delta$. By analytic considerations of Hodge subbundles, this is equivalent to saying that $\gamma_t^{p,q} = (\gamma^{p,q})_t$ for all $t \in S$.

1.5 The word ‘‘general’’ in this paper will have the following meaning. Given a family of varieties $\{V_t\}_{t \in \mathcal{T}}$ parameterized by some variety \mathcal{T} , V_t is general if t is a general point of \mathcal{T} , i.e. t belongs to the complement of a *countable* union of proper subvarieties of \mathcal{T} , governed by certain properties.

2. Proof of Theorem 0.1

Suppose to the contrary that

$$(2.0) \quad r_{3,1} : \text{CH}^3(W_t \times Y, 1) \rightarrow \{\text{Prim}^2(W_t, \mathbb{R}) \otimes H_{\text{tr}}^2(Y, \mathbb{R})\} \cap H^{2,2}(W_t \times Y),$$

is nontrivial for a general $t \in \Gamma$. Then by a standard argument (see [C-L1]), after a base change $\Gamma' \rightarrow \Gamma$ with $Z = W \times_{\Gamma} \Gamma'$, there exists $\xi \in \text{CH}^3(Z_U \times Y, 1)$ such that $r_{3,1}(\xi_t) \neq 0$ when ξ is restricted to general $t \in U$, where U is a nonempty Zariski open subset of Γ' and $Z_U = \bigcup_{t \in U} Z_t$. As in [C-L1], the key here is to extend ξ to all of Γ' ; once that is done, we apply a monodromy argument. Let $\xi = \sum_{\alpha} (f_{\alpha}, D_{\alpha})$ and $\bar{\xi} = \sum_{\alpha} (\bar{f}_{\alpha}, \bar{D}_{\alpha})$ be the closure of ξ in $Z \times Y$, i.e., \bar{D}_{α} is the closure of D_{α} in $Z \times Y$ and \bar{f}_{α} is just f_{α} on \bar{D}_{α} via the identification $\mathbb{C}(D_{\alpha})^{\times} = \mathbb{C}(\bar{D}_{\alpha})^{\times}$. Then

$$\text{div}(\bar{\xi}) = \sum_{\alpha} \text{div}_{\bar{D}_{\alpha}}(\bar{f}_{\alpha}),$$

is supported on $Z_B \times Y$, where $B = \Gamma' \setminus U$. Thus for $t \in U$, $\bar{\xi}_t = \xi_t$, i.e. when $\bar{\xi}$, ξ are restricted to $Z_t \times Y$. By our assumption on W_t , for each $b \in B$,

$$(2.1) \quad \begin{aligned} \text{CH}^2(Z_b \times Y; \mathbb{Q}) &= \text{CH}^0(Z_b) \otimes \text{CH}^2(Y; \mathbb{Q}) \\ &\oplus \text{CH}^1(Z_b; \mathbb{Q}) \otimes \text{CH}^1(Y; \mathbb{Q}) \\ &\oplus \text{CH}^2(Z_b; \mathbb{Q}) \otimes \text{CH}^0(Y; \mathbb{Q}) \end{aligned}$$

So after replacing $\bar{\xi}$ by $\bar{\xi} + \sum_{b \in B} (g_b, S_b)$ where $S_b \subset Z_b \times Y$, $g_b \in \mathbb{C}(S_b)^{\times}$, we may write

$$(2.2) \quad \text{div}(\bar{\xi}) = R + S + T$$

where $[R] \in \text{CH}^0(Z_B; \mathbb{Q}) \otimes \text{CH}^2(Y; \mathbb{Q})$, $[S] \in \text{CH}^1(Z_B; \mathbb{Q}) \otimes \text{CH}^1(Y; \mathbb{Q})$ and $[T] \in \text{CH}^2(Z_B; \mathbb{Q}) \otimes \text{CH}^0(Y; \mathbb{Q})$. Since $\text{div}(\bar{\xi}) \sim_{\text{rat}} 0$ on $Z \times Y$, each of R , S and T is

rationally equivalent to zero in $\text{CH}^3(Z \times Y)$. (To see this, we need only pull back $\text{div}(\bar{\xi})$ to $Z \times \{p\}$ for any $p \in Y$ to deduce that $T \sim_{\text{rat}} 0$. To show that $S \sim_{\text{rat}} 0$, use the fact that $\text{CH}^1(Y; \mathbb{Q}) \simeq \mathbb{Q}$.) From the projection $\pi_{\Gamma'} : Z \times Y \rightarrow \Gamma' \times Y$, we see that $R = \pi_{\Gamma'}^* L$ for some $L \subset \Gamma' \times Y$ and $L = 0$ in $\text{CH}^3(\Gamma' \times Y; \mathbb{Q})$. Therefore there exists $\sum_{\beta} (g_{\beta}, M_{\beta})$ with $M_{\beta} \subset \Gamma' \times Y$, $\dim M_{\beta} = 1$ and $g_{\beta} \in \mathbb{C}(M_{\beta})^{\times}$ such that

$$(2.3) \quad \sum_{\beta} \text{div}_{M_{\beta}}(g_{\beta}) = L$$

Therefore, if we replace $\bar{\xi}$ by $\bar{\xi} - \sum_{\beta} (\pi_{\Gamma'}^* g_{\beta}, \pi_{\Gamma'}^* M_{\beta})$, we have $\text{div}(\bar{\xi}) = S + T$ and $r_{3,1}(\bar{\xi}_t)$ unchanged for a general $t \in \Gamma'$. Similarly, from the projection $\pi_Z : Z \times Y \rightarrow Z$, we see that $T = \pi_Z^* L$ for some $L \subset Z$ and $L = 0$ in $\text{CH}^3(Z; \mathbb{Q})$. So there exists $\sum_{\beta} (g_{\beta}, M_{\beta})$ with $M_{\beta} \subset Z$ such that (2.3) holds. Again, if we replace $\bar{\xi}$ by $\bar{\xi} - \sum_{\beta} (\pi_Z^* g_{\beta}, \pi_Z^* M_{\beta})$, we have $\text{div}(\bar{\xi}) = S$ and $r_{3,1}(\bar{\xi}_t)$ unchanged for a general $t \in \Gamma$. Finally, we have to get rid of the contribution S . Fixing a smooth irreducible curve $C \subset Y$, we may assume that S is supported on $Z_B \times C$ since $\text{CH}^1(Z_B; \mathbb{Q}) \otimes \text{CH}^0(C; \mathbb{Q}) = \text{CH}^1(Z_B; \mathbb{Q}) \otimes \text{CH}^1(Y; \mathbb{Q})$, as $\text{CH}^1(Y; \mathbb{Q}) \simeq \mathbb{Q}$. Since $\text{CH}^2(Z; \mathbb{Q}) \otimes \text{CH}^0(C; \mathbb{Q}) \hookrightarrow \text{CH}^3(Z \times Y; \mathbb{Q})$ is an injection, $S = 0$ in $\text{CH}^2(Z \times C; \mathbb{Q})$. So there exists $\eta = \sum_{\beta} (g_{\beta}, M_{\beta})$ with $M_{\beta} \subset Z \times C$ such that $\text{div}(\eta) = S$. If we replace $\bar{\xi}$ by $\bar{\xi} - \eta$, we have $\text{div}(\bar{\xi}) = 0$. Of course, we have to show that $r_{3,1}(\eta_t) = 0$ for $t \in \Gamma$ general. Obviously, $\eta_t \in \text{CH}^2(Z_t \times C, 1)$ for t general. Then $r_{3,1}(\eta_t) = 0$ follows from the following diagram:

$$(2.4) \quad \begin{array}{ccc} \text{CH}^2(Z_t \times C, 1) & \rightarrow & \text{CH}^3(Z_t \times Y, 1) \\ \downarrow & & \downarrow \\ H^{1,1}(Z_t \times C, \mathbb{R}) & \rightarrow & H^{2,2}(Z_t \times Y, \mathbb{R}) \\ \downarrow & & \downarrow \\ H^{1,1}(Z_t, \mathbb{R}) \otimes H^0(C, \mathbb{R}) & \rightarrow & H^{1,1}(Z_t, \mathbb{R}) \otimes H^{1,1}(Y, \mathbb{R}) \end{array}$$

Here the bottom two vertical arrows are given by Hodge-Künneth projections. The final step in our construction is a mixed Hodge structure argument. Notice that $\bar{\xi}$ restricts to a class $\text{CH}^3(U_Z \times Y, 1)$ where U_Z is the smooth set of Z . The singular set $\Sigma_Z := Z \setminus U_Z$ is easily seen to be a finite number of points, as a result of possible ramification of $\Gamma' \rightarrow \Gamma$ over the singular set $\Sigma \subset \Gamma$ of $f : W \rightarrow \Gamma$. By applying the existence of the cycle class map to Deligne cohomology (see [Blo2]), together with the appropriate exact sequence in [E-V] (Cor. 2.10), we arrive at the Betti image¹

$$\text{CH}^3(U_Z \times Y, 1) \rightarrow H^5(U_Z \times Y, \mathbb{R}),$$

which is the result of the composite

$$\text{CH}^3(U_Z \times Y, 1) \rightarrow H_{\mathcal{D}}^5(U_Z \times Y, \mathbb{R}(3)) \rightarrow H^5(U_Z \times Y, \mathbb{R}),$$

where $H_{\mathcal{D}}^{\bullet}$ is Deligne cohomology. This map factors accordingly as follows:

$$\text{CH}^3(U_Z \times Y, 1) \rightarrow H_{\mathcal{D}}^5(U_Z \times Y, \mathbb{Q}(3)) \rightarrow F^3 H^5(U_Z \times Y, \mathbb{Q}) \rightarrow H^5(U_Z \times Y, \mathbb{R}),$$

where

$$F^3 H^5(U_Z \times Y, \mathbb{Q}) := F^3 H^5(U_Z \times Y, \mathbb{C}) \cap H^5(U_Z \times Y, \mathbb{Q}).$$

¹We are purposely ignoring Tate twists here.

Let

$$[\Delta_Y] = \oplus_{p+q=4} [\Delta_Y(p, q)] \in \oplus_{p+q=4} H^p(Y, \mathbb{Q}) \otimes H^q(Y, \mathbb{Q}) = H^4(Y \times Y, \mathbb{Q}),$$

be the Künneth decomposition of the diagonal class of Y . One can argue that the $\Delta_Y(p, q)$'s are algebraic cycles. We can modify $\bar{\xi} \in \text{CH}^3(U_Z \times Y, 1)$ by

$$\{\text{Id}_{U_Z} \otimes \Delta_Y(2, 2)\}_* \bar{\xi},$$

so that now the Betti image of $\bar{\xi}$ in $H^5(U_Z \times Y, \mathbb{R})$ lies in $H^3(U_Z, \mathbb{Q}) \otimes H^2(Y, \mathbb{Q})$. Next, by the nature of the singular set $\Sigma_Z \subset Z$ (see the Appendix), observe that

$$F^3(H^3(U_Z, \mathbb{Q}) \otimes H^2(Y, \mathbb{Q})) \subset H^3(U_Z, \mathbb{Q}) \otimes \mathbb{Q}[H_Y],$$

where $[H_Y] \in H^2(Y, \mathbb{Q})$ is a hyperplane class. The primitive projection $H^2(Y, \mathbb{Q}) \rightarrow \text{Prim}^2(Y, \mathbb{Q}) = \mathbb{Q}[H_Y]^\perp$ is induced by an algebraic cycle $w \in \text{CH}^2(Y \times Y; \mathbb{Q})$. [Explicitly $w = \{\Delta_Y - \frac{1}{(H_Y^2)_Y} H_Y \times H_Y\}$.] If we further modify $\bar{\xi}$ by $(\text{Id}_{U_Z} \otimes \{w\})_* \bar{\xi}$, then the corresponding Betti image of $\bar{\xi}$ in $H^5(U_Z \times Y, \mathbb{R})$ is zero. We now summarize what we have arrived at thus far. We have constructed a class $\bar{\xi} \in \text{CH}^2(U_Z \times Y, 1)$ with the property that for $t \in U$, $r_{3,1}(\bar{\xi}_t)$ is given by the restriction $r_{3,1}(\bar{\xi})|_{Z_t \times Y}$, i.e. induced by the restriction

$$\{H^2(U_Z, \mathbb{R}) \otimes H^2(Y, \mathbb{R})\} \cap F^2 H^4(U_Z \times Y, \mathbb{C})$$

↓

$$\{H^2(Z_t, \mathbb{R}) \otimes H^2(Y, \mathbb{R})\} \cap H^{2,2}(Z_t \times Y)$$

↓

$$\{\text{Prim}^2(Z_t, \mathbb{R}) \otimes H_{\text{tr}}^2(Y, \mathbb{R})\} \cap H^{2,2}(Z_t \times Y).$$

This is a result of the exact sequence (see [E-V])

$$\frac{H^4(U_Z \times Y, \mathbb{R})}{\{v + \bar{v} \mid v \in F^3 H^4(U_Z \times Y, \mathbb{C})\}} \rightarrow H_{\mathbb{D}}^5(U_Z \times Y, \mathbb{R}(3)) \rightarrow H^5(U_Z \times Y, \mathbb{R}).$$

Now recall the notion of horizontal displacement in 1.4 of section 1.

LEMMA 2.5. *Let $\tilde{Z} \supset U_Z$ be a smooth projective compactification of U_Z , and for $t \in U$, let $j_t : Z_t \times Y \hookrightarrow \tilde{Z} \times Y$ be the inclusion. Then there is a class $R \in \{H^2(\tilde{Z}, \mathbb{R}) \otimes H^2(Y, \mathbb{R})\} \cap H^{2,2}(\tilde{Z} \times Y)$ such that $r_{3,1}(\xi_t) = j_t^*(R)$ for all $t \in U$. In particular, as $t \in U$ varies, the class $r_{3,1}(\bar{\xi}_t)$ varies by horizontal displacement with the property that its Hodge (p, q) components also deform horizontally.*

PROOF. The latter statement is a well known consequence of the former. For the former statement, we will use the fact that $H^2(U_Z, \mathbb{Q}) \rightarrow H^2(Z_t, \mathbb{Q})$ is a morphism of mixed Hodge structures. It is likewise well known that the lowest weight $W_2 H^2(U_Z, \mathbb{Q})$ is given by the image $H^2(\tilde{Z}, \mathbb{Q}) \rightarrow H^2(U_Z, \mathbb{Q})$. Now since $t \in U$, $H^2(Z_t, \mathbb{Q})$ has a pure Hodge structure of weight 2. Thus, by a weight argument, the image of the composite

$$H^2(\tilde{Z}, \mathbb{Q}) \rightarrow H^2(U_Z, \mathbb{Q}) \rightarrow H^2(Z_t, \mathbb{Q}),$$

coincides with the image of

$$H^2(U_Z, \mathbb{Q}) \rightarrow H^2(Z_t, \mathbb{Q}).$$

Lemma 2.5 easily follows from this. \square

In summary, as $t \in U$ varies, the class $r_{3,1}(\bar{\xi}_t)$ varies by a *horizontal* displacement whose Hodge (p, q) components displace *horizontally*, i.e. preserving the given Hodge type. But over the set where $\Gamma' \rightarrow \Gamma$ is unramified, one can find open sets $\Delta_{\Gamma'} \subset U \subset \Gamma'$, $\Delta \subset \Gamma$, in the strong topology, such that $\Delta_{\Gamma'} \simeq \Delta$. Thus $r_{3,1}(\bar{\xi}_t) = 0$, by virtue of the fact that we are dealing with a product of surfaces $Z_t \times Y$, with Y fixed, together with:

LEMMA 2.6. *Consider a 1-parameter family $\{W_t\}_{t \in \Gamma}$ of surfaces given by a morphism $f : W \rightarrow \Gamma$, with smooth set $U \subset \Gamma$. Assume that $\text{Prim}^{2,0}(W_t) \neq 0$ for $t \in U$ and that for fixed $t_0 \in U$, the monodromy group action $\pi_1(U) \rightarrow \text{Aut}(\text{Prim}^2(W_{t_0}))$ is irreducible. Let $\Delta \subset U$ be a disk, and assume given $\gamma_t \in \text{Prim}^2(W_t, \mathbb{C})$, a horizontal displacement of a class γ for $t \in \Delta$. If the (p, q) components of γ_t also horizontally displace, then $\gamma_t = 0$.*

PROOF. This follows from a standard monodromy argument, together with the analyticity of Hodge subbundles. \square

Thus we have arrived at a contradiction of our assumption in (2.0), and hence the proof of Theorem 0.1 is complete.

3. Proof of Proposition 0.3

The following conjecture stated in [L], can be thought of as a variant of the Bloch-Beilinson conjecture:

CONJECTURE 3.0. *Let $V/\bar{\mathbb{Q}}$ be a smooth quasiprojective variety. Then the Abel-Jacobi map $\Phi_{k, \mathbb{Q}} : \text{CH}_{\text{hom}}^k(V/\bar{\mathbb{Q}}; \mathbb{Q}) \rightarrow J^k(V(\mathbb{C})) \otimes \mathbb{Q}$ is injective.*

Here the definition of the Abel-Jacobi map for smooth quasiprojective varieties, which is an extension of Griffiths' prescription, involves Carlson's extension class interpretation of intermediate jacobians ([Ca]). A detailed description of this map for example can be found in [Ja2] (§9). We now make use of the following result:

THEOREM 3.1. ([L]) *Assume given a smooth projective variety X . Then for all k , there is a filtration*

$$\begin{aligned} \text{CH}^k(X; \mathbb{Q}) &= F^0 \supset F^1 \supset \dots \supset F^\ell \supset F^{\ell+1} \\ &\supset \dots \supset F^k \supset F^{k+1} = F^{k+2} = \dots, \end{aligned}$$

which satisfies the following

- (i) $F^1 = \text{CH}_{\text{hom}}^k(X; \mathbb{Q})$
- (ii) $F^2 \subset \ker \Phi_{k, \mathbb{Q}} : \text{CH}_{\text{hom}}^k(X; \mathbb{Q}) \rightarrow J^k(X) \otimes \mathbb{Q}$.
- (iii) $F^\ell \bullet F^r \subset F^{\ell+r}$, where \bullet is the intersection product.

(iv) F^ℓ is preserved under push-forwards f_* and pull-backs f^* , where $f : X \rightarrow Y$ is a morphism of smooth projective varieties. [In short, F^ℓ is preserved under the action of correspondences between smooth projective varieties.]

(v) $Gr_F^\ell := F^\ell / F^{\ell+1}$ factors through the Grothendieck motive. More specifically, let us assume that the Künneth components of the diagonal class $[\Delta] = \bigoplus_{p+q=2n} [\Delta(p, q)] \in H^{2n}(X \times X, \mathbb{Q})$ are algebraic. Then

$$\Delta(2n - 2k + r, 2k - r)_* \Big|_{Gr_F^\ell \text{CH}^k(X; \mathbb{Q})} = \begin{cases} \text{Identity} & \text{if } r = \ell \\ 0 & \text{otherwise} \end{cases}$$

- (vi) Let $D^k(X) := \bigcap_\ell F^\ell$. If Conjecture 3.0 above holds, then $D^k(X) = 0$.

The reader may wish to consult [Ja2] (§11) and [Mu] for the connection between the above theorem and the general conjectures of Bloch/Beilinson/Murre on this subject. Now consider a general choice of 1-parameter family of $K3$ surfaces $\{W_t\}_{t \in \Gamma}$, as well as a general choice of $K3$ surface Y . We can assume that if W_t is singular for a given $t \in \Gamma$, then its singular set is a single ordinary double point. In this case let \tilde{W}_t be its minimal desingularization. One can easily argue that a Künneth decomposition as in (0.2) of Theorem 0.1 for $\mathrm{CH}^2(\tilde{W}_t \times Y; \mathbb{Q})$ implies a corresponding Künneth decomposition for $\mathrm{CH}^2(W_t \times Y; \mathbb{Q})$. Next, one can argue that $H^1(\tilde{W}_t) = H^3(\tilde{W}_t) = 0$ and that for all $t \in \Gamma$:

$$\{H^2(\tilde{W}_t, \mathbb{Q}) \otimes H^2(Y, \mathbb{Q})\} \cap H^{2,2}(\tilde{W}_t \times Y) = H_{\mathrm{alg}}^2(\tilde{W}_t, \mathbb{Q}) \otimes H_{\mathrm{alg}}^2(Y, \mathbb{Q}).$$

Thus we are reduced to proving the following:

LEMMA 3.2. *Let X and Y be smooth projective surfaces with the property that $H^j(X, \mathbb{Q}) = H^j(Y, \mathbb{Q}) = 0$ for odd j , and that*

$$\{H^2(X, \mathbb{Q}) \otimes H^2(Y, \mathbb{Q})\} \cap H^{2,2}(X \times Y) = H_{\mathrm{alg}}^2(X, \mathbb{Q}) \otimes H_{\mathrm{alg}}^2(Y, \mathbb{Q}).$$

If Conjecture 3.0 holds, then

$$\mathrm{CH}^2(X \times Y; \mathbb{Q}) = \sum_{i=0}^2 \mathrm{CH}^i(X; \mathbb{Q}) \otimes \mathrm{CH}^{2-i}(Y; \mathbb{Q}).$$

PROOF. According to Theorem 3.1, together with Conjecture 3.0, we have for any smooth projective variety V for which the diagonal Δ admits an algebraic Künneth decomposition:

$$\begin{aligned} \mathrm{CH}^k(V; \mathbb{Q}) &= \bigoplus_{\ell=0}^k \Delta(2 \dim V - 2k + \ell, 2k - \ell)_* \mathrm{CH}^k(V; \mathbb{Q}) \\ &= \bigoplus_{\ell=0}^k Gr_F^\ell \mathrm{CH}^k(V; \mathbb{Q}). \end{aligned}$$

Since X and Y are surfaces, and that the respective diagonals Künneth decompose algebraically, the same holds true for the diagonal of $X \times Y$ (see [K1]). Since the odd cohomologies vanish, we have that

$$(3.3) \quad \mathrm{CH}^2(X \times Y; \mathbb{Q}) = Gr_F^0 \mathrm{CH}^2(X \times Y; \mathbb{Q}) \oplus Gr_F^2 \mathrm{CH}^2(X \times Y; \mathbb{Q}).$$

Note that

$$\begin{aligned} Gr_F^0 \mathrm{CH}^2(X \times Y; \mathbb{Q}) &\subset H^4(X \times Y, \mathbb{Q}) \cap H^{2,2}(X \times Y) \\ &\subset H^4(X, \mathbb{Q}) \otimes H^0(Y, \mathbb{Q}) \oplus \\ &\quad H_{\mathrm{alg}}^2(X, \mathbb{Q}) \otimes H_{\mathrm{alg}}^2(Y, \mathbb{Q}) \oplus H^0(X, \mathbb{Q}) \otimes H^4(Y, \mathbb{Q}) \end{aligned}$$

Thus $Gr_F^0 \mathrm{CH}^2(X \times Y; \mathbb{Q})$ Künneth decomposes. By (3.3) it suffices to show that $Gr_F^2 \mathrm{CH}^2(X \times Y; \mathbb{Q})$ Künneth decomposes. But

$$Gr_F^2 \mathrm{CH}^2(X \times Y; \mathbb{Q}) = \Delta(6, 2)_* \mathrm{CH}^2(X \times Y; \mathbb{Q}),$$

which by Theorem 3.1(i), depends only on the cohomology class

$$[\Delta(6, 2)] \in \{H^6(X \times Y, \mathbb{Q}) \otimes H^2(X \times Y, \mathbb{Q})\} \cap H^{4,4}(X \times Y \times X \times Y).$$

The components of this cohomology class lie in:

$$\begin{aligned}
\text{(I)} &: H^4(X, \mathbb{Q}) \otimes H_{\text{alg}}^2(Y, \mathbb{Q}) \otimes H_{\text{alg}}^2(X, \mathbb{Q}) \otimes H^0(Y, \mathbb{Q}) \\
\text{(II)} &: H_{\text{alg}}^2(X, \mathbb{Q}) \otimes H^4(Y, \mathbb{Q}) \otimes H^0(X, \mathbb{Q}) \otimes H_{\text{alg}}^2(Y, \mathbb{Q}) \\
\text{(III)} &: \{H^4(X, \mathbb{Q}) \otimes H^2(Y, \mathbb{Q}) \otimes H^0(X, \mathbb{Q}) \otimes H^2(Y, \mathbb{Q})\} \\
&\quad \cap H^{4,4}(X \times Y \times X \times Y) \\
\text{(IV)} &: \{H^2(X, \mathbb{Q}) \otimes H^4(Y, \mathbb{Q}) \otimes H^2(X, \mathbb{Q}) \otimes H^0(Y, \mathbb{Q})\} \\
&\quad \cap H^{4,4}(X \times Y \times X \times Y)
\end{aligned}$$

It is easy to check that all terms of $[\Delta(6, 2)]$ in (I) - (IV) are algebraic. For (I) and (II), this is obvious. For (III), the cycle is given by $\text{Pr}_{24}^*(\Delta_Y(2, 2)) \cap \text{Pr}_{13}^*(\Delta_X(4, 0))$, and for (IV), the cycle is given by

$$\text{Pr}_{13}^*(\Delta_X(2, 2)) \cap \text{Pr}_{24}^*(\Delta_Y(4, 0)),$$

where Pr_{ij} is the projection of $X \times Y \times X \times Y$ onto the ij -th factor. It is now an elementary exercise to show that the description of these terms implies the Künneth decomposition for $Gr_F^2 \text{CH}^2(X \times Y; \mathbb{Q})$. \square

4. Appendix

Our purpose here is to correct a mistake in [C-L1] and give an alternative approach to our (first) main result Theorem 0.1. In [C-L1], an alternative proof for the triviality of regulator maps on a very general surface in \mathbb{P}^3 of degree $d \geq 5$ (the content of Theorem 1.1, op. cit.) was given, among other things. The precise statement is this. For a smooth complex projective surface Z , let

$$\underline{cl}_{2,1} : \text{CH}^2(Z, 1) \rightarrow \frac{H^2(Z, \mathbb{C})}{F^2 H^2(Z, \mathbb{C}) + \text{Hg}^1(Z) \otimes \mathbb{C} + H^2(Z, \mathbb{Q}(2))},$$

be the reduced cycle class map (such as can be found in in [MS1]), where $\text{Hg}^1(Z)$ is the Hodge group (and where we include twists). “Sufficiently general” will have the same meaning as in 1.5 of section 1.

THEOREM 4.1. (*Theorem 1.1 in [C-L1]*) *For a sufficiently general surface $Z \subset \mathbb{P}^3$ of degree $d \geq 5$, the map $\underline{cl}_{2,1}$ is trivial.*

For completeness and for the convenience of the reader, we also state:

THEOREM 4.2. (*Theorem 1.2 in [C-L1]*) *Let $X = C_1 \times C_2$ be a product of two general curves (resp. hyperelliptic curves), where the genus $g(C_1) \geq 1$ and $g(C_2) \geq 2$. Then $\underline{cl}_{2,1}$ is trivial.*

The arguments for Theorem 0.1 here and other results in that paper are modelled after that proof of Theorem 1.1 in [C-L1]. Before sketching that proof and correcting the mistake, it is essential to explain the “yoga” of our construction in [C-L1] and which is also apparent in this present work. Suppose we are given a smooth and proper morphism $\rho : W \rightarrow T$ of smooth quasiprojective varieties. For $t \in T$, put $W_t = \rho^{-1}(t)$. Now consider a higher Chow cycle $\xi \in \text{CH}^k(W, 1)$, which of course restricts to a class $\xi_t \in \text{CH}^k(W_t, 1)$. Note that the real regulator takes its values $r_{k,1}(\xi_t) \in H^{k-1, k-1}(W_t, \mathbb{R})$. This is precisely due to the fact that W_t is a smooth projective variety. More specifically $r_{k,1}(\xi_t) \in H_{\mathcal{D}}^{2k-1}(W_t, \mathbb{R}(k))$ (real Deligne cohomology), but by a weight argument, and ignoring twists, $H_{\mathcal{D}}^{2k-1}(W_t, \mathbb{R}(k)) \simeq H^{k-1, k-1}(W_t, \mathbb{R})$. A similar statement for the image of $r_{k,1}(\xi)$ cannot be made, since W is not assumed complete, albeit

smooth (the smoothness of W is required to get a cycle class map to cohomology). Let us assume for the moment that W is projective (and still smooth). Note that $r_{k,1}(\xi_t)$, being the restriction of $r_{k,1}(\xi)$ to W_t , arises from the restriction map $H^{k-1,k-1}(W, \mathbb{R}) \rightarrow H^{k-1,k-1}(W_t, \mathbb{R})$. This latter restriction translates to Betti rigidity, which implies that the class $r_{k,1}(\xi_t)$ varying with $t \in T$ is given by horizontal displacement, for which the Hodge (p, q) components also displace horizontally (which follows from the restriction map being a morphism of Hodge structures). In some cases W need not be complete to arrive at Betti rigidity, this being the import of our discussion in section 2, leading up to Lemma 2.5. Once Betti rigidity is established, a monodromy argument is used to show the triviality of a reduced or modified version of the real regulator for a general member of a specific family of varieties in question. To summarize our approach, a higher Chow cycle on a general smooth projective variety is spread out and extended to a cycle $\xi \in \text{CH}^k(W, 1)$, where W_t is now our general member for a general choice of $t \in T$. In section 2, our choice of W is $W = U_Z \times Y$. For a suitably modified ξ we establish Betti rigidity, and then employ the monodromy argument in Lemma 2.6. Alternatively, we could extend ξ to a class $\bar{\xi} \in \text{CH}^k(\bar{W}, 1)$, where \bar{W} is a *smooth* compactification of W . Then $r_{k,1}(\bar{\xi}) \in H^{k-1,k-1}(\bar{W}, \mathbb{R})$ restricts to $r_{k,1}(\xi_t) \in H^{k-1,k-1}(W_t, \mathbb{R})$, viz., Betti rigidity. It is this latter approach that we employed in the proof of Theorem 1.1 of [C-L1], and for which we will now explain below.

Here is a sketch of that proof. For starters, we choose a general pencil $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ of surfaces of degree $d \geq 5$. After a base change $\Gamma \rightarrow \mathbb{P}^1$, there exists a class $\xi \in \text{CH}^2(Y_U, 1)$ where $Y = X \times_{\mathbb{P}^1} \Gamma$, $U \subset \Gamma$ is a nonempty Zariski open set in Γ and $Y_U = \cup_{b \in U} Y_b$. Using an argument similar to that of Theorem 0.1, we show that ξ can be extended to all of Y . The key here is that $\text{CH}^1(X_t)_{\mathbb{Q}} = \mathbb{Q}$ for every $t \in \mathbb{P}^1$, which follows from a result of M. Green on Noether-Lefschetz loci.

Note that Y might be singular at a point p which is a rational double point (node) of the fiber Y_t . Locally at p , Y is given by $x^2 + y^2 + z^2 + t^\mu = 0$, where μ is the ramification index of $Y \rightarrow X$ at p . The singularity of Y at p can be resolved by a sequence of blowups $\tilde{Y} = Y^{(m)} \rightarrow Y^{(m-1)} \rightarrow \dots \rightarrow Y^{(0)} = Y$ over p , where $m = \lfloor \mu/2 \rfloor$. It was proved that ξ can be extended to a class $\tilde{\xi} \in \text{CH}^2(\tilde{Y}, 1)$. Then a standard monodromy argument, as used here, concluded that the image of $\tilde{\xi}$, under the regulator map, is trivial. However, the argument there that ξ can be extended on \tilde{Y} was flawed. Using the localization sequence, it is easy to see that the obstruction to extending ξ is the map

$$\text{CH}^1(Q) \rightarrow \text{CH}^2(\tilde{Y})$$

where Q is the exceptional divisor of $Y^{(m)} \rightarrow Y^{(m-1)}$. It can be extended if and only if the above map is injective. If μ is odd, Q is a cone over a smooth conic curve and hence $\text{CH}^1(Q) = \mathbb{Q}$; and the injectivity of $\text{CH}^1(Q) \rightarrow \text{CH}^2(\tilde{Y})$ follows. If μ is even, $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and this where the trouble originates. A flawed argument was given in [C-L1], where it was wrongly assumed that \tilde{Y} can be blown down along each ruling of Q with the resulting variety (only analytic *a priori*) projective. Indeed, it is a classical example that the small resolutions of a quartic threefolds with nodes are not projective (not even Kähler). It is a subtle problem to determine whether $\text{CH}^1(Q) \rightarrow \text{CH}^2(\tilde{Y})$ is injective for a singular threefold Y with nodes, where \tilde{Y} is the blowup of Y at the nodes with Q the exceptional divisor. Take e.g. Hironaka's example of a nonsingular complete threefold which is Moishezon but not projective. Let X be a nonsingular projective threefold and $C \subset X$ be an irreducible curve on X with a node. Let Y be the blowup of X along C . Then Y has a node p over the

node of C . Let \tilde{Y} be the blowup of Y at p and Z be the blowdown of \tilde{Y} along one of the two rulings of the exceptional divisor $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ of $\tilde{Y} \rightarrow Y$. Then Z is Moishezon but not projective. We also see in this example that the two rulings of Q are linearly independent in $\mathrm{CH}^2(\tilde{Y})$, i.e., $\mathrm{CH}^1(Q) \rightarrow \mathrm{CH}^2(\tilde{Y})$ is injective if and only if C is rational. Hence some extra efforts are needed to justify the injectivity of $\mathrm{CH}^1(Q) \rightarrow \mathrm{CH}^2(\tilde{Y})$ for our particular Y . Basically, we will prove the following:

PROPOSITION 4.3. *Let $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a very general pencil of surfaces of degree $d \geq 2$ and let $\Gamma \rightarrow \mathbb{P}^1$ be a finite map from a smooth curve Γ to \mathbb{P}^1 and $Y = X \times_{\mathbb{P}^1} \Gamma$. Then for any desingularization $\tilde{Y} \rightarrow Y$ with exceptional locus Q , $\mathrm{CH}^1(Q) \rightarrow \mathrm{CH}^2(\tilde{Y})$ is an injection.*

Let $|\mathcal{O}_{\mathbb{P}^3}(d)|$ be the complete linear series of surfaces of degree d in \mathbb{P}^3 and $B \cong \mathbb{P}^2 \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ be a two-dimensional sublinear series constructed as follows.

We choose $V \subset \mathbb{P}^1 \times \mathbb{P}^3$ to be a general pencil of quadrics in \mathbb{P}^3 and fix a general surface $S \subset \mathbb{P}^3$ of degree $d - 2$. We let $W \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a pencil of surfaces of degree d whose fiber $R_t = V_t \cup S$ for every $t \in \mathbb{P}^1$. Let $b \in \mathbb{P}^1$ be a point where V_t is singular, i.e., a cone over a smooth conic with vertex p . This family W/\mathbb{P}^1 has the following important property at p .

After an arbitrary base change $\Gamma \rightarrow \mathbb{P}^1$, we obtain $\mathcal{W} = W \times_{\mathbb{P}^1} \Gamma$. At a point $q \in \mathcal{W}$ that lies over p , \mathcal{W} is locally given by $x^2 + y^2 + z^2 + t^\mu = 0$, where μ is the ramification index of the map $\mathcal{W} \rightarrow W$ at q . In the case that $\mu > 1$, \mathcal{W} is singular at q , where the singularity can be resolved by a sequence of $m = \lfloor \mu/2 \rfloor$ blowups: $\tilde{\mathcal{W}} = \mathcal{W}^{(m)} \rightarrow \mathcal{W}^{(m-1)} \rightarrow \dots \rightarrow \mathcal{W}^{(0)} = \mathcal{W}$. Let $E^{(i)}$ be the exceptional divisor of $\mathcal{W}^{(i)} \rightarrow \mathcal{W}^{(i-1)}$. It is not hard to see that $E^{(i)}$ is a cone over a conic curve for $\mu > 2i$ and $E^{(i)} \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $\mu = 2i$, i.e., $i = m$ and $\underline{\mu} = 2m$ being even. Let $E = E_1 + E_2 + \dots + E_m$ be the exceptional divisor of $\tilde{\mathcal{W}} \rightarrow \mathcal{W}$, where E_i is the proper transform of $E^{(i)}$. So $E_i \cong \mathbb{F}_2$ is the blowup of $E^{(i)}$ at the vertex of the cone for $i < m$ and $Q = E_m \cong E^{(m)}$ is a cone over a conic curve if μ is odd and $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ if μ is even. Also note that $E_i \cap E_{i+1}$ and $E_i \cap E_{i-1}$ are two disjoint sections of $E_i \cong \mathbb{F}_2$ over \mathbb{P}^1 . We claim that the map

$$\mathrm{CH}^1(E) \hookrightarrow \mathrm{CH}^2(\tilde{\mathcal{W}})$$

is an injection. It is not hard to see that $\mathrm{CH}^1(E_1 \cup E_2 \cup \dots \cup E_{m-1}) \hookrightarrow \mathrm{CH}^2(\tilde{\mathcal{W}})$ is an injection. Hence $\mathrm{CH}^1(E) \hookrightarrow \mathrm{CH}^2(\tilde{\mathcal{W}})$ is injective if and only if

$$\mathrm{CH}^1(Q) \hookrightarrow \mathrm{CH}^2(\tilde{\mathcal{W}})$$

is. This is more or less obvious when Q is a cone over a conic curve, i.e., when μ is odd. If μ is even, $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ as mentioned above; the two rulings l_1 and l_2 of Q are linearly independent in $\mathrm{CH}^2(\tilde{\mathcal{W}})$ since in this case $\tilde{\mathcal{W}}$ can be blown down along l_i with the resulting (analytic) variety projective (hence Kähler).

Let $B = \mathbb{P}^2 \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ and $X \subset B \times \mathbb{P}^3$ be a two-dimensional linear system of degree d surfaces that contains the pencil W . We choose X to be general with this property. Let $C \cong \mathbb{P}^1 \subset B$ be the line such that $W = X \times_B C$. Let

$$B_\Delta = \overline{\{t \in B - C : X_t \text{ singular}\}}$$

Due to our general choice of X , C meets B_Δ transversely at some point $b \in C \cap B_\Delta$ where V_b is singular at p . For simplicity, we may blow up B at a general point on C and realize B as a pencil, which we still denote by B . So $B \cong \mathbb{F}_1$ is a \mathbb{P}^1 bundle over \mathbb{P}^1 , where we parameterize the base by s and let $B_0 = C$.

We assume that there is a class $\xi \in \text{CH}^2(Y_U, 1)$ after a suitable base change $\phi : D \rightarrow B$, where $Y = X \times_B D$, U is a nonempty open set of D and $Y_U = X \times_B U$. Our purpose is to show that for $s \in \mathbb{P}^1$ general, ξ can be extended over Y_s after we desingularize Y over B_Δ . However, this is hard to prove directly. Our strategy is to find another base change $F \rightarrow B$ and replace D by $D \times_B F$.

For simplicity, we assume that $\phi : D \rightarrow B$ is simply ramified along $B_\Delta - b$ with multiplicity μ locally at b . (By simple ramification, we mean ramification over a smooth divisor.) If the preimage $\phi^{-1}(b)$ of b is not unique, we just do the following for every point in $\phi^{-1}(b)$. We let $F \rightarrow B$ be a finite map simply ramified along B_Δ with multiplicity μ locally at b . It is helpful to write down the local equations of D and F .

Let $\Delta_{s,t}^2$ be an analytic neighborhood of $b \in B$, where $B_0 = C = \{s = 0\}$ and $B_\Delta = \{t = 0\}$ locally. Then D and F are locally given by

$$w^\mu + s^\alpha t + f(s, t)t^2 = 0$$

and

$$w^\mu + t = 0,$$

respectively. Also note that X is

$$x^2 + y^2 + z^2 = t$$

locally at p .

Note that $D \times_B F$ is not normal if $\mu > 1$. Let G be an irreducible component of $D \times_B F$ after normalization. So we have the diagram

$$\begin{array}{ccc} & G & \\ & \swarrow & \searrow \\ D & & F \\ & \searrow & \swarrow \\ & B & \end{array}$$

The important thing here is that the map $G \rightarrow F$ is not ramified over the locus $\{u = 0\}$.

Let $Z = X \times_B F$. Obviously, Z has singularities of type $x^2 + y^2 + z^2 = u^\mu$ over B_Δ , which can be resolved by a sequence of blowups: $\tilde{Z} = Z^{(m)} \rightarrow Z^{(m-1)} \rightarrow \dots \rightarrow Z^{(0)} = Z$ with $m = \lfloor \mu/2 \rfloor$. Let $E = E_1 + E_2 + \dots + E_m$ be the exceptional divisor of $\tilde{Z} \rightarrow Z$, where E_i is the proper transform of the exceptional divisor of $Z^{(i)} \rightarrow Z^{(i-1)}$. Let $Q = E_m$. Then due to our choice of the pencil $C = B_0 \subset B$, $\text{CH}^1(Q_0) \hookrightarrow \text{CH}^2(\tilde{Z}_0)$ is an injection. Hence $\text{CH}^1(Q_s) \hookrightarrow \text{CH}^2(\tilde{Z}_s)$ is injective for $s \in \mathbb{P}^1$ general.

Let $\Sigma = \tilde{Z} \times_F G$. The class $\xi \in \text{CH}^2(Y_U, 1)$ can be lifted to Σ such that it is defined over an open set of G . We still denote the class by ξ . Let $P = \varphi^{-1}(Q)$, where φ is the map $\Sigma \rightarrow \tilde{Z}$. For $s \in \mathbb{P}^1$, we have the diagram

$$\begin{array}{ccc} \text{CH}^1(P_s) & \rightarrow & \text{CH}^2(\Sigma_s) \\ \downarrow & & \downarrow \\ \text{CH}^1(Q_s) & \rightarrow & \text{CH}^2(\tilde{Z}_s) \end{array}$$

Since $G \rightarrow F$ is not ramified over B_Δ , P_s is a disjoint union of copies of Q_s for $s \in \mathbb{P}^1$ general. Consequently, $\mathrm{CH}^1(P_s) \hookrightarrow \mathrm{CH}^2(\Sigma_s)$ is an injection since $\mathrm{CH}^1(Q_s) \hookrightarrow \mathrm{CH}^2(\tilde{Z}_s)$ is. So ξ can be extended on Σ_s over $\varphi^{-1}(E)$ for s general. And since $G \rightarrow F$ is not ramified over B_Δ , Σ_s is smooth along $\varphi^{-1}(E)$ for s general. This is all we need when we apply the regulator map.

Now let us go back to the proof of Theorem 0.1. After we extend ξ to all of Z , an alternative approach is to desingularize Z and use the argument here to extend ξ to \tilde{Z} . There is no change to the rest of the proof.

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