1. Arithmetic Genus

For a projective scheme $X \subset \mathbb{P}^n$, let

\begin{equation}
\varphi_X(m) = \chi(O_X(m)) = \sum_{i \geq 0} (-1)^i h^0(O_X(m))
\end{equation}

be the Hilbert polynomial of $X$. We call

\begin{equation}
\tag{1.2} p_a(X) = (-1)^{\dim X} (\varphi_X(0) - 1) = (-1)^{\dim X} \left( \sum_{i \geq 0} (-1)^i h^0(O_X) - 1 \right)
\end{equation}

the arithmetic genus of $X$. Note that $p_a(X)$ does not depend on the embedding of $X$ in $\mathbb{P}^n$.

When $X$ is integral, $h^0(O_X) = 1$. Then

\begin{equation}
\tag{1.3} p_a(X) = (-1)^{\dim X+1} (h^1(O_X) - h^2(O_X) + \ldots).
\end{equation}

In the cases that $\dim X = 1$, $p_a(X)$ is simply $h^1(O_X)$.

For a plane curve $C \subset \mathbb{P}^2$ of degree $d$, we have computed its Hilbert polynomial

\begin{equation}
\tag{1.4} \chi(O_C(m)) = md - \left(\frac{d-1}{2}\right) + 1.
\end{equation}

Also $h^0(O_C) = 1$ by the exact sequence

\begin{equation}
\tag{1.5} 0 \rightarrow O(-d) \rightarrow O \rightarrow O_C \rightarrow 0
\end{equation}

and the vanishing $H^i(O_{\mathbb{P}^2}(-d)) = 0$ for $i = 0, 1$. So its arithmetic genus is

\begin{equation}
\tag{1.6} p_a(C) = (-1)(h^0(O_C) - h^1(O_C) - 1) = h^1(O_C)
\end{equation}

\begin{equation}
= (-1)(\chi(O_C(0)) - 1) = \left(\frac{d-1}{2}\right).
\end{equation}

By Serre Duality, $H^1(O_C) \cong H^0(\omega_C) = H^0(O_C(d-3))$, where $\omega_C$ is the dualizing sheaf of $C$.

**Theorem 1.1** (Serre Duality For Local Complete Intersections). Let $X$ be a projective scheme in $P = \mathbb{P}^n$. If $X$ is a local complete intersection of codimension $r$ in $P$, i.e., $I_X$ is locally generated by $r$ elements $f_1, f_2, \ldots, f_r$ for $r = n - \dim X$, then

\begin{equation}
\tag{1.7} H^i(E)^\vee \cong H^{n-r-i}(\omega_X \otimes E^\vee)
\end{equation}

for all vector bundles $E$ on $X$ and all $i$, where $\omega_X$ is the dualizing sheaf of $X$ given by

\begin{equation}
\omega_X \cong K_P \otimes (\wedge^r I_X/I_X^2)^{-1}.
\end{equation}
In particular, if $X$ is smooth, $\omega_X \cong K_X$. If $X$ is a complete intersection in $P$ of type $(d_1, d_2, \ldots, d_r)$, then

$$I_X/I_X^2 \cong \bigoplus_{i=1}^{r} O_X(-d_i)$$

and hence

$$\omega_X \cong O_X(d_1 + d_2 + \ldots + d_r - n - 1).$$

So $p_a(C) = h^0(K_C)$ for a smooth projective curve $C$.

2. Geometric Genus and $\delta$-Invariant

Let $C$ be an integral projective curve. We call $g(C) = p_a(C^\nu)$ the geometric genus of $C$, where $\nu: C^\nu \to C$ is the normalization of $C$.

**Theorem 2.1.** For an integral projective curve $C$, $g(C) \leq p_a(C)$.

**Proof.** The normalization $\nu: C^\nu \to C$ gives a short exact sequence

$$0 \to O_C \to \nu_* O_{C^\nu} \to \nu_* O_{C^\nu}/O_C \to 0$$

Taking the long exact sequence, we obtain

$$0 \to H^0(O_C) \to H^0(\nu_* O_{C^\nu}) \to H^0(\nu_* O_{C^\nu}/O_C) \to 0$$

$$\to H^1(O_C) \to H^1(\nu_* O_{C^\nu}) \to H^1(\nu_* O_{C^\nu}/O_C)$$

$$H^1(O_{C^\nu}) \cong 0$$

The isomorphism $H^1(\nu_* O_{C^\nu}) \cong H^1(O_{C^\nu})$ follows from Leray spectral sequence. So we have

$$h^0(\nu_* O_{C^\nu}/O_C) = h^1(O_C) - h^1(O_{C^\nu}) = p_a(C) - g(C)$$

and hence $p_a(C) \geq g(C)$. □

The difference between $p_a(C)$ and $g(C)$ is given by

$$h^0(\nu_* O_{C^\nu}/O_C) = \sum_{p \in C_{\text{sing}}} \dim_C (\nu_* O_{C^\nu}/O_C \otimes O_p)$$

and called the total $\delta$-invariants of $C$. 

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