

Solution for Midterm II ¹

1. (a) $HX - XH = 2X$, $XY - YX = H$ and $HY - YH = -2Y$.
 (b) $\det(H) = -1$ and $\det(X) = \det(Y) = 0$.
 (c) The eigenvalues of H are 1 and -1 and the corresponding eigenvectors are $(t, 0)$ and $(0, t)$ for $t \neq 0$, respectively.
 The eigenvalue of X is 0 and the corresponding eigenvectors are $(t, 0)$ for $t \neq 0$.
 The eigenvalue of Y is 0 and the corresponding eigenvectors are $(0, t)$ for $t \neq 0$.
 (d) You may take A and B to be any two of H , X and Y or you may come up with your own examples.

2. (a)

$$\det \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = -1.$$

- (b)

$$\begin{aligned} & \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ & \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \\ & \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 \end{bmatrix} \end{aligned}$$

¹<http://www.math.ucsb.edu/~xichen/math5a00s/mid2sol.pdf>

So

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

3.

$$\begin{aligned} \begin{bmatrix} 3 & 4 & c & 3 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 4 & c & 3 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & c-3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & c-5 & 0 \end{bmatrix} \end{aligned}$$

So the system has infinitely many solution if and only if $c - 5 = 0$, i.e., $c = 5$.

4. (a) The determinant of A is the product of the eigenvalues of A . So $\det(A) = 6$.

(b) If λ is an eigenvalue of A with eigenvector \mathbf{x} , then $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow 2A\mathbf{x} = 2\lambda\mathbf{x}$. So 2λ is an eigenvalue of $2A$. So $2A$ has eigenvalues 2, 4 and 6.

(c) $\det(2A) = 2^3 \det(A) = 48$.

(d) If λ is an eigenvalue of A with eigenvector \mathbf{x} , then $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow (A + I)\mathbf{x} = (\lambda + 1)\mathbf{x}$. So $\lambda + 1$ is an eigenvalue of $A + I$. So $A + I$ has eigenvalues 2, 3 and 4.