Solution for Midterm I [1]

1. (a) By Euler formula, $e^{\pi i/2} = \cos(\pi/2) + i\sin(\pi/2) = i$. So

$$e^{e^{\pi i/2}} = e^i = \cos(1) + i\sin(1).$$

(b) Let $x_1 = x$, $x_2 = x'$ and $x_3 = x''$. The corresponding system of 1st order ODEs is

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 1 - x_1 - x_2 - x_3 \end{cases}$$

The only equilibrium point of the system is $(1, 0, 0)$.

(c) The characteristic polynomial $D^2 + aD + b$ has a double root 2000. So the general solution is $C_1 e^{2000t} + C_2 t e^{2000t}$. Since $D^2 + aD + b = (D - 2000)^2$, $a = -4000$ and $b = 4000000$.

2. (a) The intergrating factor is $I(t) = e^{\int t dt} = e^{t^2/2}$. So

$$(e^{t^2/2}x)' = te^{t^2/2}$$

and hence

$$e^{t^2/2}x = \int te^{t^2/2}dt = e^{t^2/2} + C.$$

Therefore, the general solution is $x(t) = 1 + Ce^{-t^2/2}$. Using initial condition, we obtain $C = 1$ and $x(t) = 1 + e^{-t^2/2}$.

(b) The general solution for $x'' - 4x' + 4x = 0$ is $A_1 e^{2t} + A_2 t e^{2t}$. A trial solution for $x'' - 4x' + 4x = 1$ is in the form of C. Plugging $x = C$ into the equation we obtain $C = 1/4$. So the general solution for $x'' - 4x' + 4x = 1$ is $1/4 + A_1 e^{2t} + A_2 t e^{2t}$.

(c) The general solution for $x'' + x = 0$ is $C_1 \cos t + C_2 \sin t$. Using the initial condition, we obtain $C_1 = 1$ and $C_2 = -1$. So $x(t) = \cos t - \sin t = \sqrt{2}\cos(t + \pi/4)$.

[1] http://www.math.ucsb.edu/~xichen/math5a00s/mid1msol.pdf

(d) A trial solution is in the form $C_1 t\cos t + C_2 t\sin t$. Plugging this into the equation, we have

$$C_1(-2\sin t - t\cos t) + C_2(2\cos t - t\sin t) + C_1 t\cos t + C_2 t\sin t = \cos t$$

i.e.

$$-2C_1\sin t + 2C_2\cos t = \cos t.$$

So $C_1 = 0$ and $C_2 = 1/2$. So the general solution is

$$\frac{1}{2}t\sin t + A_1\cos t + A_2\sin t.$$

Using the initial condition, we obtain

$$x(t) = \frac{1}{2}t\sin t.$$

3. (a) $(C_1 + C_2 t + C_3 t^2 + C_4 t^3)t^2 e^t + (C_5 + C_6 t + C_7 t^2 + C_8 t^3)$
   (b) $(C_1 + C_2 t)te^{-t}\cos 2t + (C_3 + tC_4)te^{-t}\sin 2t$

4. (a) $a = 0$ and $b > 0$
   (b) $b < 0$
   (c) $a > 0$ and $b > 0$
   (d) $a < 0$ and $b > 0$