PICARD'S THEOREMS

1. Picard Theorems

Casorati-Weierstrass says that the image of an analytic function at an essential singularity is dense in $\mathbb{C}$. The Picard theorems are much strengthened versions of this result.

The Little Picard Theorem (LPT) says that a nonconstant entire function misses at most one values.

Theorem 1.1 (Little Picard Theorem)**.** *Let $f(z)$ be an entire function. If $f(\mathbb{C}) \subset \mathbb{C}\backslash\{p_1, p_2\}$ for some $p_1 \neq p_2$, then f is constant.*

The Great Picard Theorem (GPT) is a direct generalization of Casorati-Weierstrass:

Theorem 1.2 (Great Picard Theorem)**.** *Let $f(z)$ be an analytic function in $\Delta^* = \{0 < |z| < 1\}$. If $f(z)$ has an essential singularity at 0, then $\mathbb{C}\backslash f(\Delta^*)$ consists of at most one point.*

2. Bloch's Theorem

Lemma 2.1. *Let f be analytic in $\Delta = \{|z| < 1\}$ satisfying that $f(0) = 0$ and $f'(0) = 1$. If $|f(z)| \leq M$ for all $z \in \Delta$, then $f(\Delta)$ contains the disk $|w| \leq (\sqrt{M+1} - \sqrt{M})^2$.*

Proof. By Schwartz's lemma, we implicitly have $M \geq 1$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Using CIF, we have $|a_n| \leq M$ for all n. Therefore,

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n = r - \frac{Mr^2}{1-r} \tag{2.1}$$

for $|z| = r < 1$. Obviously, we can maximize the RHS of (2.1) by taking

$$r = \rho = 1 - \sqrt{\frac{M}{M+1}} \tag{2.2}$$

and correspondingly, $|f(z)| \geq (\sqrt{M+1} - \sqrt{M})^2$ for $|z| = \rho$.

For all $|w| < \rho$, $|f(z) - (f(z) - w)| = |w| \leq |f(z)|$ for all $|z| = \rho$. Therefore, $f(z) - w$ and $f(z)$ have the same number of zeros in $|z| < \rho$. It follows that $f(\Delta)$ contains the disk $|w| \leq (\sqrt{M+1} - \sqrt{M})^2$. □

Obviously, by "scaling", we have the following:

Lemma 2.2. *Let f be an analytic function on $D = \{|z-a| < R\}$. If $|f(z) - f(a)| \le M$ for all $z \in D$, then $f(D)$ contains the disk $|w - f(a)| \le (\sqrt{M + |f'(a)R|} - \sqrt{M})^2$.*

Lemma 2.3. *An analytic function $f(z)$ on Δ is 1-1 if $|f'(z) - M| < |M|$ for all $z \in \Delta$ and a constant $M \in \mathbb{C}$.*

Proof. Let z_1 and z_2 be two distinct points in Δ and let γ be the line joining z_1 and z_2. Then

$$\begin{aligned} |f(z_1) - f(z_2)| &= \left| \int_\gamma f'(z) dz \right| \\ &\ge \left| \int_\gamma M dz \right| - \left| \int_\gamma (f'(z) - M) dz \right| \\ &\ge |M| \int_\gamma |dz| - \int_\gamma |f'(z) - M||dz| > 0 \end{aligned} \tag{2.3}$$

and hence $f(z)$ is 1-1. □

Theorem 2.4 (Bloch's Theorem)**.** *Let $f(z)$ be an analytic function on Δ satisfying $f'(0) = 1$. Then there is a positive constant B (called Bloch's constant), independent of f, such that there exists a disk $S \subset \Delta$ where f is 1-1 and whose image $f(S)$ contains a disk of radius B. In particular, $B > 1/72$.*

Proof. Obviously, it is enough to show this for $f'(z)$ bounded on Δ.

Let

$$m(r, g) = \max_{|z|=r} |g(z)|. \tag{2.4}$$

We let $0 \le r_0 < 1$ be the largest number such that $(1 - r_0) m(r_0, f') = 1$. Such r_0 exists since $f'(z)$ is bounded on Δ.

Then $(1 - r) m(r, f') < 1$ for $r > r_0$ and hence

$$|f'(z)| \le \frac{1}{1 - |z|} \tag{2.5}$$

for $|z| \ge r_0$. And by principle of maximum modulus, we have

$$|f'(z)| \le m(r_0, f') = \frac{1}{1 - r_0} \tag{2.6}$$

for $|z| \le r_0$. In conclusion,

$$|f'(z)| \le \frac{1}{1 - \max(r_0, |z|)} \tag{2.7}$$

for all $z \in \Delta$.

Let $a \in \Delta$ be a number such that $|a| = r_0$ and $|f'(a)| = 1/(1 - r_0)$.

For $0 < \rho < r_0$ and $|z - a| \le \rho$, we have

$$|f'(z) - f'(a)| \le \frac{1}{1 - r_0} + \frac{1}{1 - r_0 - \rho} \tag{2.8}$$

and hence

$$|f'(z) - f'(a)| \le \frac{|z-a|}{\rho}\left(\frac{1}{1-r_0} + \frac{1}{1-r_0-\rho}\right) \tag{2.9}$$

by Schwartz's lemma. Therefore, $|f'(z) - f'(a)| < |f'(a)|$ for z in the disk

$$S = \left\{ |z-a| < \frac{\rho(1-r_0-\rho)}{2(1-r_0)-\rho} \right\}. \tag{2.10}$$

By Lemma 2.3, f is 1-1 on S. Obviously, the radius of S is maximized when we set $\rho = (2-\sqrt{2})(1-r_0)$ and correspondingly,

$$S = \left\{ |z-a| < (3-2\sqrt{2})(1-r_0) \right\}. \tag{2.11}$$

Moreover, since

$$|f(z) - f(a)| \le \ln\left(\frac{\sqrt{2}+1}{2}\right) \tag{2.12}$$

for $z \in S$ by (2.7), we conclude that $f(S)$ contains a disk of radius

$$\left(\sqrt{\ln\left(\frac{\sqrt{2}+1}{2}\right) + (3-2\sqrt{2})} - \sqrt{\ln\left(\frac{\sqrt{2}+1}{2}\right)}\right)^2 > \frac{1}{72}. \tag{2.13}$$

□

Remark 2.5. The key to the proof of Bloch's theorem is the existence of $a \in \Delta$ and positive constants C_1 and C_2 such that $|f'(z)| \le C_2|f'(a)|$ for all $|z-a| \le C_1/|f'(a)|$.

3. Proof of Picard Theorems

Obviously, the Great Picard implies the Little Picard: if an entire function $f(z)$ has an essential singularity at ∞, then the statement follows from the Great Picard; otherwise, $f(z)$ is a polynomial and it follows from the Fundamental Theorem of Algebra. Despite this, we want to prove Little Picard first since its proof provides the necessary technique for the generalization to Great Picard.

The basic idea of the proof is the following: given an entire function missing two values, we can "cook up" a function missing infinitely many values, distributed in such a way contradicting Bloch's theorem.

Suppose that $f(z)$ is an entire function satisfying $f(\mathbb{C}) \subset \mathbb{C} - \{p_1, p_2\}$ for some $p_1 \ne p_2$. Replacing $f(z)$ by $(f(z)-p_1)/(p_2-p_1)$, we may assume that $f(\mathbb{C}) \subset \mathbb{C} - \{0, 1\}$.

Since $f(z)$ is a nowhere vanishing analytic function on $\mathbb{C}$, which is simply connected, we can find $g(z)$, analytic on $\mathbb{C}$, such that

$$f(z) = \exp(g(z)). \tag{3.1}$$

Since $1 \notin f(D)$, $g(z)$ cannot assume the values $2n\pi i$ for $n \in \mathbb{Z}$. That is,

$$\frac{g(z)}{2\pi i} \notin \mathbb{Z} \tag{3.2}$$

for all $z \in \mathbb{C}$. In particular,

$$\frac{g(z)}{2\pi i} \neq 0, 1 \tag{3.3}$$

for all $z \in \mathbb{C}$. Again, using the simple connectedness of $\mathbb{C}$, there exist entire functions $\phi_1(z)$ and $\phi_2(z)$ such that

$$(\phi_1(z))^2 = \frac{g(z)}{2\pi i} \text{ and } (\phi_2(z))^2 = \frac{g(z)}{2\pi i} - 1. \tag{3.4}$$

Finally, since $\phi_1(z) - \phi_2(z) \neq 0$ on $\mathbb{C}$, we can find an entire function $\sigma(z)$ such that

$$\exp(\sigma(z)) = \phi_1(z) - \phi_2(z). \tag{3.5}$$

In summary, we can put (3.1), (3.4) and (3.5) together

$$\begin{aligned} f(z) &= \exp(g(z)), \\ (\phi_1(z))^2 &= \frac{g(z)}{2\pi i}, \\ (\phi_2(z))^2 &= \frac{g(z)}{2\pi i} - 1 \text{ and} \\ \exp(\sigma(z)) &= \phi_1(z) - \phi_2(z) \end{aligned} \tag{3.6}$$

which shows how to "cook up" $\sigma(z)$ from $f(z)$. A terse way to put it is

$$\sigma(z) = \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right) \tag{3.7}$$

where log and $\sqrt{\ }$ should be regarded as multi-valued functions.

It follows from (3.4) that $(\phi_1(z))^2 - (\phi_2(z))^2 = 1$. Combining with (3.5), we obtain

$$\exp(-\sigma(z)) = \phi_1(z) + \phi_2(z). \tag{3.8}$$

Thus, $2\phi_1(z) = \exp(\sigma(z)) + \exp(-\sigma(z))$ and hence

$$\frac{g(z)}{2\pi i} = \frac{\exp(2\sigma(z)) + \exp(-2\sigma(z))}{4} + \frac{1}{2} \tag{3.9}$$

Since the left hand side of (3.9) misses out all integers, it is easy to see that

$$\sigma(z) \notin \Lambda = \left\{\pm \ln(\sqrt{n+1} + \sqrt{n}) + \frac{m\pi}{2} i : n \in \mathbb{Z}^+ \text{ and } m \in \mathbb{Z}\right\} \tag{3.10}$$

for all $z \in \mathbb{C}$. Clearly, Λ is the vertices of rectangles covering $\mathbb{C}$, with each rectangle of size at most

$$\ln(1 + \sqrt{2}) \times \frac{\pi}{2} \tag{3.11}$$

Therefore, these rectangles have diagonals at most

$$\sqrt{(\ln(1+\sqrt{2}))^2 + \left(\frac{\pi}{2}\right)^2} < 2 \tag{3.12}$$

and $\sigma(\mathbb{C}) \subset \mathbb{C}\backslash\Lambda$ does not contain any disk of radius ≥ 1. On the other hand, $\sigma(z)$ is a nonconstant entire function. Hence by Bloch's theorem, $\sigma(\mathbb{C})$ contains disks of arbitrarily large radius. Contradiction.

The construction (3.6) works for all simply connected open set D and all analytic functions $f(z)$ on D with $f(D) \subset \mathbb{C} - \{0, 1\}$. Therefore, we can find $\sigma(z)$, analytic on D, such that $\sigma(D) \subset \mathbb{C}\backslash\Lambda$. Then the derivative $\sigma'(p)$ at a point $p \in D$ has an upper bound

$$|\sigma'(p)| \leq \frac{1}{Br_p} \text{ for } r_p = \inf_{q\in\partial D} |p - q| \tag{3.13}$$

where B is the Bloch constant and r_p is the largest number such that the disk $\{|z - p| < r_p\} \subset D$.

This line of argument leads to the following:

Theorem 3.1. *Let D be a connected open set and $\{f_n\}$ be a sequence of analytic functions on D satisfying $f_n(D) \subset \mathbb{C} - \{0, 1\}$. Suppose that there exists a point $p \in D$ such that*

$$\lim_{n\to\infty} f_n(z_0) = w_0 \notin \{0, 1\} \tag{3.14}$$

Then $\{f_n\}$ is a normal family.

Proof. Let us first prove it for D simply connected.

For every f_n, we can construct σ_n satisfying (3.6). Then we see by (3.13) that $\{\sigma_n'\}$ is uniformly bounded on every compact subset of D and hence a normal family. If in addition, $\{\sigma_n(z_0)\}$ is bounded, then we can conclude that $\{\sigma_n\}$ is a normal family.

After removing a branch locus $\gamma \subset \mathbb{C}$, we can define a branch $g(z)$ of

$$\log\left(\sqrt{\frac{\log(z)}{2\pi i}} - \sqrt{\frac{\log(z)}{2\pi i} - 1}\right) \tag{3.15}$$

on $\mathbb{C}\backslash\gamma$. Since $w_0 \neq 0, 1$, we may choose the curve γ such that $w_0 \notin \gamma$. For example, we may take γ be the union of two disjoint rays, originating from 0 and 1, respectively. Then $g(w_0)$ is well defined and we may choose σ_n such that

$$\lim_{n\to\infty} \sigma_n(z_0) = g(w_0). \tag{3.16}$$

Therefore, $\{\sigma_n(z_0)\}$ is bounded and $\{\sigma_n'\}$ is uniformly bounded on every compact subset of D. Consequently, $\{\sigma_n\}$ and hence $\{f_n\}$ are normal.

For an arbitrary connected open set D, we can find a countable open covers of D by simply connected sets:

$$D = \bigcup_{n=0}^{\infty} D_n \tag{3.17}$$

For example, we may simply take D_n to be the open disks contained in D with rational centers and radii. We may order D_n in such a way that

$$z_0 \in D_0 \text{ and } D_n \cap D_{n+1} \neq \emptyset \text{ for all } n \in \mathbb{N}. \tag{3.18}$$

Then there exists a subsequence $\{f_{0n}\} \subset \{f_n\}$ converging on D_0. Since $\lim f_{0n}(z_0) = w_0 \notin \{0, 1\}$, there exists a point $z_1 \in D_0 \cap D_1$ such that

$$\lim_{n\to\infty} f_{0n}(z_1) = w_1 \notin \{0, 1\} \tag{3.19}$$

So there exists a subsequence $\{f_{1n}\} \subset \{f_{0n}\}$ converging on D_1. Inductively, we obtain

$$\{f_n\} \supset \{f_{0n}\} \supset \{f_{1n}\} \supset ... \tag{3.20}$$

such that $\{f_{mn}\}$ converges on D_m for all m. Then we take the diagonal $\{f_{nn}\}$, which converges on $D = \cup D_m$. □

Now we are ready to prove Great Picard Theorem. Suppose that $f(z)$ is an analytic function on $D = \{0 < |z| < 1\}$ such that $f(z)$ has an essential singularity at 0 and $f(D) \subset \mathbb{C} - \{0, 1\}$.

By Casorati-Weierstrass, for every $w \in \mathbb{C}$, there exists a sequence $c_n \in D$ such that $\lim c_n = 0$ and $\lim f(c_n) = w$. Let us choose $w = 2$. Hence we have

$$\lim_{n\to\infty} c_n = 0 \text{ and } \lim_{n\to\infty} f(c_n) = 2 \neq 0, 1. \tag{3.21}$$

Let us choose $|c_n| < 1/2$ for all n and consider

$$f_n(z) = f(2c_n z) \tag{3.22}$$

for $n = 1, 2, ...$ Obviously, $f_n(z)$ is analytic on D, $f_n(D) \subset \mathbb{C} - \{0, 1\}$ and

$$\lim_{n\to\infty} f_n\left(\frac{1}{2}\right) = \lim_{n\to\infty} f(c_n) = 2 \neq 0, 1. \tag{3.23}$$

Then by Theorem 3.1, $\{f_n\}$ is a normal family. For simplicity, let us assume that $f_n(z)$ converges to an analytic function $g(z)$ on D.

Let $K = \{|z| = 1/2\}$. Since f_n converges uniformly to g on K, $f_n(z)$ is uniformly bounded on K. That is,

$$\max_{z\in K} |f_n(z)| \leq M \tag{3.24}$$

for all n and some constant M. That is,

$$\max_{|z|=|c_n|} |f(z)| = \max_{z\in K} |f_n(z)| \leq M. \tag{3.25}$$

By Maximum Modulus,

$$\max_{|c_n|\leq|z|\leq 1/2} |f(z)| \leq \max\left(M, \max_{|z|=1/2} |f(z)|\right). \tag{3.26}$$

And since $\lim c_n = 0$, this implies that $f(z)$ is uniformly bounded on $\{|z| \leq 1/2\}$. By Riemann Extension, $f(z)$ has a removable singularity at 0. Contradiction.