1. Rouche’s Theorem

The way to apply Rouche’s Theorem for a holomorphic function $g(z)$ on $D$ is to find a “dominant” term $f(z)$ in $g(z)$ such that

\begin{equation}
|f(z)| > |f(z) - g(z)|
\end{equation}

on the boundary of $D$.

**Proposition 1.1.** Let $a$ be a complex number satisfying $|a| > 5/2$. Show that the power series

\begin{equation}
F(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{n^2}}
\end{equation}

defines an entire function which does not vanish on the boundary of the annulus

\begin{equation}
|a^{2n-2}| < |z| < |a^{2n}|
\end{equation}

and has exactly one zero inside the annulus for $n = 1, 2, \ldots$.

**Proof.** Since the radius of convergence of the power series (1.2) is

\begin{equation}
\lim_{n \to \infty} \sqrt[n]{|a^{n^2}|} = \lim_{n \to \infty} |a^n| = \infty
\end{equation}

$F(z)$ is entire.

We claim that

\begin{equation}
\left| \frac{z^n}{a^{n^2}} \right| > \left| F(z) - \frac{z^n}{a^{n^2}} \right|
\end{equation}

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for $|z| = |a^{2n}|$. This follows from

$$
| F(z) - \frac{z^n}{a^{n^2}} | = \left| \sum_{k=0}^{n-1} \frac{z^k}{a^{k^2}} + \sum_{k=n+1}^{\infty} \frac{z^k}{a^{k^2}} \right|
\leq \sum_{k=0}^{n-1} |a|^{2nk-k^2} + \sum_{k=n+1}^{\infty} |a|^{2nk-k^2}
\leq \sum_{k=0}^{n-1} |a|^{n^2-3n+2+3k} + \sum_{k=n+1}^{\infty} |a|^{n^2+3n+2-3k}
\leq \frac{|a|^{n^2-1}}{1-|a|^{-3}} + \frac{|a|^{n^2-1}}{1-|a|^{-3}} < |a|^2 = \left| \frac{z^n}{a^{n^2}} \right|
$$

for $|z| = |a^{2n}|$ and $|a| > 5/2$.

Therefore, $F(z)$ has no zeros on $\{|z| = |a^{2n}|\}$ and it has exactly $n$ zeros in $\{|z| < |a^{2n}|\}$ by Rouche’s Theorem. □

2. Growth of Entire Functions

One important fact about a holomorphic function $f(z)$ is that the $n$-th derivatives of $f(z)$ is given by Cauchy integral formula

$$
f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-z)^{n+1}} dw
$$

for $|z| < R$.

**Proposition 2.1.** For an entire function $f(z)$, we let

$$
M(r) = \max_{|z| \leq r} |f(z)|.
$$

Let $f(z)$ be an entire function with

$$
\limsup_{r \to \infty} \frac{\log M(r)}{r} = l.
$$

Show that the infinite series

$$
F(z) = \sum_{n=0}^{\infty} f^{(n)}(z)
$$

converges if $l < 1$ and diverges if $l > 1$.

**Proof.** Suppose that $l < 1$. So there exists $\lambda < 1$ such that $|f(z)| \leq ce^{\lambda |z|}$ for some constant $c > 0$ and all $z$. By Cauchy Integral Formula,

$$
|f^{(n)}(z)| = \left| \frac{n!}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-z)^{n+1}} dw \right| \leq \frac{c(n!)R}{(R - |z|)^{n+1}} e^{\lambda R}
$$

for all $n \in \mathbb{N}$.\n
We fix $r > 0$ and want to show that
\begin{equation}
\sum_{n=0}^{\infty} |f^{(n)}(z)| < \infty
\end{equation}
in $\{ |z| \leq r \}$. We choose $R = r + \lambda^{-1} n$. Then
\begin{equation}
|f^{(n)}(z)| \leq ce^{\lambda r} \left( 1 + \frac{\lambda r}{n} \right) \lambda^n e^n \frac{n!}{n^n}
\end{equation}
for all $n \geq 1$ and $|z| \leq r$ by (2.5). So it suffices to show the convergence of the series
\begin{equation}
\sum_{n=1}^{\infty} ce^{\lambda r} \left( 1 + \frac{\lambda r}{n} \right) \lambda^n e^n \frac{n!}{n^n} = \sum_{n=1}^{\infty} a_n
\end{equation}
which follows from the ratio test:
\begin{equation}
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \lambda e \left( \frac{n^n}{(n+1)^n} \right) = \lambda < 1.
\end{equation}
When $l > 1$, if $F(z)$ converges for $z = 0$, then
\begin{equation}
\lim_{n \to \infty} f^{(n)}(0) = 0 \Rightarrow |f^{(n)}(0)| \leq c
\end{equation}
for a constant $c$ and all $n$. Then
\begin{equation}
|f(z)| = \left| \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \right| \leq \sum_{n=0}^{\infty} \frac{c|z|^n}{n!} = ce^{|z|}
\end{equation}
which contradicts
\begin{equation}
\limsup_{r \to \infty} \frac{\log M(r)}{r} = l > 1.
\end{equation}

**Proposition 2.2.** Let $f(z)$ be an entire function with $M(r)$ defined in (2.2). Show that if there is a constant $0 < \alpha < 1$ such that
\begin{equation}
\lim_{r \to \infty} \frac{M(\alpha r)}{M(r)} > 0,
\end{equation}
then $f(z)$ is a polynomial and the above limit is $\alpha^n$ with $n = \deg f$.

**Proof.** Since the limit (2.13) exists, there exists a constant $c > 0$ such that $M(\alpha r) \geq c M(r)$ for all $r \geq 1$. Therefore,
\begin{equation}
M(\alpha^n r) \geq c^n M(r) \Rightarrow c^{-n} M(1) \geq M(\alpha^{-n}).
\end{equation}
By Cauchy Integral Formula, we have
\begin{equation}
|f^{(n)}(0)| = \left| \frac{m!}{2\pi i} \int_{|z| = \alpha^{-n}} \frac{f(z)}{z^{m+1}} dz \right| \leq (m!) M(\alpha^{-n}) \alpha^{mn}
\end{equation}
\begin{equation}
\leq (m!) M(1) \left( \frac{\alpha^n}{c} \right)^n
\end{equation}
for all $m$ and $n$. Then for all $m$ satisfying $\alpha^m < c$, $f^{(m)}(0) = 0$ by taking $n \to \infty$ in (2.15). Therefore, $f(z)$ is a polynomial.

If $f(z)$ is a polynomial of degree $n$, then

\begin{equation}
\lim_{z \to \infty} \frac{f(z)}{z^n} = c \Rightarrow \lim_{r \to \infty} \frac{M(r)}{r^n} = c \Rightarrow \lim_{r \to \infty} \frac{M(\alpha r)}{M(r)} = \alpha^n.
\end{equation}

\[ \Box \]

3. Conformal Equivalence

We say that two open sets $D_1$ and $D_2$ in $\mathbb{C}$ are conformally equivalent if there exists a biholomorphic map $f : D_1 \to D_2$. Conformally equivalent open sets must be homeomorphic but the converse does not hold.

**Proposition 3.1.** Show that $D = \{1 < |z| < 2\}$ and $\Delta^* = \{0 < |z| < 1\}$ are not conformally equivalent.

**Proof.** Suppose that there is a biholomorphic map $f : \Delta^* \to D$. Since $f(\Delta^*) = D$, $|f(z)| < 2$ for all $z \in \Delta^*$. By Riemann extension theorem, $f$ can be extended to $f : \Delta \to \mathbb{C}$. Since $f$ is continuous, $f(\Delta)$ lies in the closure $\overline{D}$ of $D$. On the other hand, by open mapping theorem, $f(\Delta)$ is open in $\mathbb{C}$. So $f(\Delta)$ is an open set satisfying

\begin{equation}
D \subset f(\Delta) \subset \overline{D}.
\end{equation}

Since $D$ is the interior of $\overline{D} = \{1 \leq |z| \leq 2\}$, we must have $f(\Delta) = D$. That is, $f(0) = q \in D$.

Let $p_1 = 0$. Since $f : \Delta^* \to D$ is one-to-one, there exists $p_2 \neq p_1$ such that $f(p_2) = q$. Let $U_1 = \{|z - p_1| < \varepsilon\}$ and $U_2 = \{|z - p_2| < \varepsilon\}$ for $\varepsilon > 0$ satisfying $U_1 \cap U_2 = \emptyset$ and $U_1, U_2 \subset \Delta$. By open mapping theorem, $f(U_j) = V_j$ are open sets containing $q$ for $j = 1, 2$. So $V_1 \cap V_2$ contains an open disc centered at $q$. Therefore, there is $q' \neq q$ such that $q' \in V_1 \cap V_2$. Then $f^{-1}(q')$ contains at least two points $p'_1 \in U_1$ and $p'_2 \in U_2$ with $p'_1 \neq p'_2$. Let $p'_1, p'_2 \neq 0$. This contradicts the hypothesis that $f : \Delta^* \to D$ is one-to-one. \[ \Box \]