(1) Let $f_1(z)$ and $f_2(z)$ be two analytic functions on $D = \{|z| < 1\}$. Suppose that $f_1(0) = f_2(0)$, $f_2$ is biholomorphic and $f_1(D) \subset f_2(D)$. Show that $|f_1'(0)| \leq |f_2'(0)|$.

Find a necessary and sufficient condition for the equality to hold.

(2) Let $f(z)$ be a holomorphic function on $D = \{|z| < 1\}$. If $f(0) = 0$, show that the series

$$\sum_{n=1}^{\infty} f(z^n)$$

uniformly converges on every compact subset of $D$.

(3) Compute the integral

$$\int_0^\infty \frac{dx}{1 + x^r}$$

for some $r > 1$.

(4) Let $a$ be a complex number satisfying $|a| > 5/2$. Show that the power series

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{n^2}}$$

defines an entire function which does not vanish on the boundary of the annulus $|a^{2n-2}| < |z| < |a^{2n}|$ and has exactly one zero inside the annulus for $n = 1, 2, \ldots$.

(5) For an entire function $f(z)$, we let

$$M(r) = \max_{|z| \leq r} |f(z)|.$$ 

Let $f(z)$ be an entire function with

$$\limsup_{r \to \infty} \frac{\log M(r)}{r} = l.$$ 

Show that the infinite series

$$F(z) = \sum_{n=0}^{\infty} f^{(n)}(z)$$

converges if $l < 1$ and diverges if $l > 1$.

(6) Let $f(z)$ be an entire function with $M(r)$ defined in the previous problem. Show that if there is a constant $0 < \alpha < 1$ such that

$$\lim_{r \to \infty} \frac{M(\alpha r)}{M(r)} > 0,$$

then $f(z)$ is a polynomial and the above limit is $\alpha^n$ with $n = \deg f$. 

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(7) Let $f(z)$ be an analytic function on $\{|z| < 1\}$. If $f(0) = 0$ and $|f(z)| < 1$ for all $z \in D$, show that

$$|f''(0)| \leq 2 - 2|f'(0)|^2.$$ 

Hint: Apply Schwartz’s Lemma to the function

$$g(z) = \frac{z - 1}{f(z)}$$

for $g(z) = z^{-1}f(z)$. 