ON A THEOREM OF G. POLYA

Theorem 0.1. *Suppose that* $\{a_n : n = 0, 1, ...\}$ *be a sequence of complex numbers such that*

$$\sum_{n=0}^{\infty} \left| \frac{a_{n+1}}{a_n} \right|^2 \tag{0.1}$$

converges. Then the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{0.2}$$

converges everywhere and

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^2} \tag{0.3}$$

converges, where $z_1, z_2, ..., z_n, ...$ *are the zeros of the entire function* $f(z)$.

Lemma 0.2 (Schur)**.** *Let* $A = [a_{ij}]_{n\times n}$ *be an* $n \times n$ *matrix with complex entries and let* $\lambda_1, \lambda_2, ..., \lambda_n$ *be the eigenvalues of* A*, counted with multiplicities. Then*

$$|\lambda_1|^2 + |\lambda_2|^2 + ... + |\lambda_n|^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2. \tag{0.4}$$

Proof. We observe that

$$\mathrm{Tr}(\overline{A}^T A) = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2 \tag{0.5}$$

and

$$\mathrm{Tr}(\overline{A}^T A) = \mathrm{Tr}(\overline{\overline{U}^T A U}^T \, \overline{U}^T A U) \tag{0.6}$$

for all unitary matrices U. Then (0.4) follows easily if we diagonalize A using unitary matrices. □

Lemma 0.3. *Let* $f_n(z)$ *be a sequence of analytic functions in an open neighborhood of* $\{|z| \le R\}$ *converging uniformly to* $f(z)$ *on* $\{|z| \le R\}$*. Suppose that* $f(0) \ne 0$ *and* $f(z) \ne 0$ *for* $|z| = R$*. Let* $z_{n1}, z_{n2}, ..., z_{nm}$ *be the zeros of* $f_n(z)$ *and* $z_1, z_2, ..., z_m$ *be the zeros of* $f(z)$ *in* $|z| < R$*, respectively. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{m} \frac{1}{|z_{nk}|^2} = \sum_{k=1}^{m} \frac{1}{|z_k|^2}. \tag{0.7}$$

Proof. For all analytic functions $g(z)$ in $|z| < R$, we define the function $N_g(r)$ to be the number of zeros of $g(z)$ in $|z| < r$. Then

$$\sum_{k=1}^{m} \frac{1}{|z_{nk}|^2} = \int_0^R \frac{dN_{f_n}(r)}{r^2} dr = \frac{m}{R^2} + \int_0^R \frac{2N_{f_n}(r)}{r^3} dr. \tag{0.8}$$

We define

$$\mu(r) = \min_{|z|=r} |f(z)|. \tag{0.9}$$

By Rouche's Theorem, $N_f(r) = N_{f_n}(r)$ if

$$\mu(r) > \max_{|z|=r} |f(z) - f_n(z)|. \tag{0.10}$$

Suppose that $|f(z) - f_n(z)| < \varepsilon$ for all $|z| \leq R$. Then

$$\left| \sum_{k=1}^{m} \frac{1}{|z_{nk}|^2} - \sum_{k=1}^{m} \frac{1}{|z_k|^2} \right| < 2m \int_{\mu(r) \leq \varepsilon} \frac{dr}{r^3}. \tag{0.11}$$

It is obvious that $\mu(r)$ is a continuous function on $[0, R]$, $\mu(0) \neq 0$ and it has only finitely many zeros on $[0, R]$. It is easy to see that

$$\lim_{\varepsilon \to 0} \int_{\mu(r) \leq \varepsilon} \frac{dr}{r^3} = 0. \tag{0.12}$$

□