Math 506 Homework 4
Due Apr. 12, 2012

We use the notation $\mathcal{O}[[z_1, z_2, ..., z_n]]$ to denote the ring of germs of analytic functions in n variables $z_1, z_2, ..., z_n$.

(1) We call

$$f(z_1, z_2, ..., z_n) = \sum_{I \in \mathbb{N}^n} a_I z^I \in \mathcal{O}[[z_1, z_2, ..., z_n]]$$

has multiplicity m at $(0, 0, ..., 0)$ if

$$m = \min\{m_1 + m_2 + ... + m_n : a_I \neq 0 \text{ for } I = (m_1, m_2, ..., m_n)\}.$$

Suppose that $f \in \mathcal{O}[[z_1, z_2]]$ has multiplicity 2 at $(0, 0)$. Show that $\mathcal{O}[[z_1, z_2]]/(f) \cong \mathcal{O}[[x, y]]/(x^2 - y^m)$ for some integer $m \geq 2$.

(2) Let $f(x, y) \in \mathcal{O}[[x, y]]$. Suppose that

$$f(x, y) = (x - c_1 y)(x - c_2 y)...(x - c_m y) + \sum_{j+k>m} a_{jk} x^j y^k$$

where $c_1, c_2, ..., c_m, a_{jk} \in \mathbb{C}$ are constants. Show that if $c_1, c_2, ..., c_m$ are m distinct numbers, then there exists $g_1(y), g_2(y), ..., g_m(y) \in \mathcal{O}[[y]]$ such that

$$f(x, y) = (x - g_1(y))(x - g_2(y))...(x - g_m(y))h(x, y)$$

where $h(x, y) \in \mathcal{O}[[x, y]]$ and $h(0, 0) \neq 0$.

(3) Let $\mathcal{O}([z])$ be the quotient field of $\mathcal{O}[[z]]$. Then any finite extension of $\mathcal{O}([z])$ is $\mathcal{O}([z])$ itself and given by $\phi : \mathcal{O}([z]) \to \mathcal{O}([z])$ with $\phi(z) = z^n$ for some $n > 0$.

(4) Let $\mathbb{C}[[z, w]]$ be the ring of formal power series. Prove Weierstrass Preparation Theorem on $\mathbb{C}[[z, w]]$. That is, let

$$f(z, w) = w^m + \sum_{(i,j) \neq (0,m)} a_{ij} z^i w^j \in \mathbb{C}[[z, w]]$$

Then $f(z, w) = g(z, w)h(z, w)$ where $g(z, w) = w^d + a_1(z)w^{d-1} + ... + a_d(z)$ is a Weierstrass polynomial and $h(0, 0) \neq 0$.

(5) Prove the Weierstrass Division Theorem on $\mathbb{C}[[z, w]]$. That is, let $g(z, w) \in \mathbb{C}[[z], w] = \mathbb{C}[[z]][w]$ be a Weierstrass polynomial of degree d in w. Then for every $f \in \mathbb{C}[[z, w]]$, we can write $f = gh + r$, where $r(z, w) \in \mathbb{C}[[z], w]$ is a polynomial of degree $< d$ in w.

(6) Show that $\mathbb{C}[[z, w]]$ is a UFD.

(7) Let $f \in \mathcal{O}[[z_1, z_2]]$ where $f \not\equiv 0$, f is irreducible and $f(0, 0) = 0$. Then $\mathcal{O}[[z_1, z_2]]/(f) \cong \mathcal{O}[[z]]$ if and only if $\mathcal{O}[[z_1, z_2]]/(f)$ is integrally closed (normal).

(8) Show that

$$\mathcal{O}[[z_1, z_2, ..., z_n]]/(z_1^2 + z_2^2 + ... + z_n^2)$$

is integrally closed for $n \geq 3$ but

$$\mathcal{O}[[z_1, z_2, ..., z_n]]/(z_1^2 + z_2^2 + ... + z_n^2) \ncong \mathcal{O}[[x_1, x_2, ..., x_{n-1}]].$$

(9) Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic map given by polynomials. If the complex Jacobian J_f has rank m at $(0, 0, ..., 0)$, then $f(\mathbb{C}^n)$ is dense in $\mathbb{C}^m$.

(10) For $m > n$, does there exist a holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^m$ such that $f(\mathbb{C}^n)$ is dense in $\mathbb{C}^m$? Hint: Consider $f : \mathbb{C} \to \mathbb{C}^2$ given by $f(z) = (\exp(2\pi \exp(2\pi z)), \exp((1 + i) \exp(z)))$.