Math 506 Homework 2
Due Mar. 8, 2012

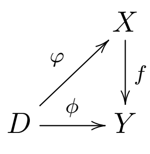
(1) Find a Riemannian metric $ds^2 = h(z)|dz|^2$ on $\mathcal{H} = \{\mathrm{Im}(z) > 0\}$ such that
$$\phi^*(ds^2) = ds^2$$
for all $\phi \in \mathrm{Aut}(\mathcal{H})$.

(2) Let f be a holomorphic map from $\Delta = \{|z| < 1\}$ to Δ. Show that
$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \le |f(z)| \le \frac{|f(0)| + |z|}{1 - |f(0)||z|}$$
for all $z \in \Delta$.

(3) Let $f(z)$ be an analytic function in $\{a < \mathrm{Re}(z) < b\}$ and let
$$M(x) = \sup_{\mathrm{Re}(z)=x} |f(z)|.$$
Show that $\log M(x)$ is a convex function on (a, b) if $f(z)$ is uniformly bounded in $\{a < \mathrm{Re}(z) < b\}$. Recall that a function $g(x)$ is convex on an interval I if
$$tg(x_1) + (1-t)g(x_2) \ge g(tx_1 + (1-t)x_2)$$
for all $x_1, x_2 \in I$ and all $0 \le t \le 1$. Here we allow $M(x) = 0$, in which cases $\log(M(x)) = -\infty$.

(4) Let $\Delta^* = \{0 < |z| < 1\}$. Find $\mathrm{Aut}(\Delta^*)$.

(5) A *covering map* $f : X \to Y$ between two topological spaces X and Y is a continuous surjective map with the property that for every $y \in Y$, there is a open neighborhood V of y such that
$$f^{-1}(V) = \bigsqcup U_\alpha$$
where U_α are open in X and $f|_{U_\alpha} : U_\alpha \to V$ is a homeomorphism for each α. And we call X *a covering space* of Y. When X is simply connected, we call f a *universal covering map* and X a (the) *universal covering space* of Y. Let $f : X \to Y$ be a holomorphic covering map between two connected open sets X and Y in $\mathbb{C}$. Then every holomorphic map $\phi : D \to Y$ from a simply connected open set $D \subset \mathbb{C}$ to Y can be lifted to a holomorphic map $D \to X$, i.e., there exists a holomorphic map $\varphi : D \to Y$ such that $\phi = f \circ \varphi$, namely, the diagram



commutes.

(6) Let $f : X \to Y$ be a holomorphic covering map between two connected open sets X and Y in $\mathbb{C}$. If Y is simply connected, then so is X and f is biholomorphic.

(7) Let $\Delta_{R,r}$ denote the annulus $r < |z| < R$ for $R > r \geq 0$. Find a holomorphic (universal) covering map $f : \mathcal{H} \to \Delta_{R,r}$.

(8) Find $\mathrm{Aut}(\Delta_{R,r})$. Hint: Let $f : \mathcal{H} \to \Delta_{R,r}$ be the covering map obtained in the previous problem. Show that every $\phi \in \mathrm{Aut}(\Delta_{R,r})$ can be lifted to some $\varphi \in \mathrm{Aut}(\mathcal{H})$, i.e, there exists $\varphi \in \mathrm{Aut}(\mathcal{H})$ such that the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\varphi} & \mathcal{H} \\ \big\downarrow f & & \big\downarrow f \\ \Delta_{R,r} & \xrightarrow{\phi} & \Delta_{R,r} \end{array}$$

commutes.

(9) For what $R > r \geq 0$ and $R' > r' \geq 0$, are $\Delta_{R,r}$ and $\Delta_{R',r'}$ conformally equivalent?

(10) Show that for every $n \in \mathbb{Z}^+$, there exists $c_n \in \mathbb{R}$ such that $|a_n| \leq c_n$ for all analytic and 1-1 functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ...$$

in Δ, where we let $a_1 = 1$.