1) Find a Riemannian metric $ds^2 = h(z)|dz|^2$ on $\mathcal{H} = \{ |\text{Im}(z)| > 0 \}$ such that

$$\phi^*(ds^2) = ds^2$$

for all $\phi \in \text{Aut}(\mathcal{H})$.

2) Let $f$ be a holomorphic map from $\Delta = \{ |z| < 1 \}$ to $\Delta$. Show that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}$$

for all $z \in \Delta$.

3) Let $f(z)$ be an analytic function in $\{ a < \text{Re}(z) < b \}$ and let

$$M(x) = \sup_{\text{Re}(z)=x} |f(z)|.$$ 

Show that $\log M(x)$ is a convex function on $(a,b)$ if $f(z)$ is uniformly bounded in $\{ a < \text{Re}(z) < b \}$. Recall that a function $g(x)$ is convex on an interval $I$ if

$$tg(x_1) + (1-t)g(x_2) \geq g(tx_1 + (1-t)x_2)$$

for all $x_1, x_2 \in I$ and all $0 \leq t \leq 1$. Here we allow $M(x) = 0$, in which cases $\log(M(x)) = -\infty$.

4) Let $\Delta^* = \{ 0 < |z| < 1 \}$. Find $\text{Aut}(\Delta^*)$.

5) A covering map $f : X \to Y$ between two topological spaces $X$ and $Y$ is a continuous surjective map with the property that for every $y \in Y$, there is a open neighborhood $V$ of $y$ such that $f^{-1}(V) = \bigsqcup U_\alpha$ where $U_\alpha$ are open in $X$ and $f|_{U_\alpha} : U_\alpha \to V$ is a homeomorphism for each $\alpha$. And we call $X$ a covering space of $Y$. When $X$ is simply connected, we call $f$ a universal covering map and $X$ a (the) universal covering space of $Y$. Let $f : X \to Y$ be a holomorphic covering map between two connected open sets $X$ and $Y$ in $\mathbb{C}$. Then every holomorphic map $\phi : D \to Y$ from a simply connected open set $D \subset \mathbb{C}$ to $Y$ can be lifted to a holomorphic map $D \to X$, i.e., there exists a holomorphic map $\varphi : D \to Y$ such that $\phi = f \circ \varphi$, namely, the diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\xleftarrow{\varphi} & \downarrow{\phi} & \\
D & \xrightarrow{\phi} & Y \\
\end{array}$$

commutes.
(6) Let $f : X \to Y$ be a holomorphic covering map between two connected open sets $X$ and $Y$ in $\mathbb{C}$. If $Y$ is simply connected, then so is $X$ and $f$ is biholomorphic.

(7) Let $\Delta_{R,r}$ denote the annulus $r < |z| < R$ for $R > r \geq 0$. Find a holomorphic (universal) covering map $f : \mathbb{H} \to \Delta_{R,r}$.

(8) Find $\text{Aut}(\Delta_{R,r})$. Hint: Let $f : \mathbb{H} \to \Delta_{R,r}$ be the covering map obtained in the previous problem. Show that every $\phi \in \text{Aut}(\Delta_{R,r})$ can be lifted to some $\varphi \in \text{Aut}(\mathbb{H})$, i.e, there exists $\varphi \in \text{Aut}(\mathbb{H})$ such that the diagram

\[
\begin{array}{ccc}
\mathbb{H} & \xrightarrow{\varphi} & \mathbb{H} \\
\downarrow{f} & & \downarrow{f} \\
\Delta_{R,r} & \xrightarrow{\phi} & \Delta_{R,r}
\end{array}
\]

commutes.

(9) For what $R > r \geq 0$ and $R' > r' \geq 0$, are $\Delta_{R,r}$ and $\Delta_{R',r'}$ conformally equivalent?

(10) Show that for every $n \in \mathbb{Z}^+$, there exists $c_n \in \mathbb{R}$ such that $|a_n| \leq c_n$ for all analytic and 1-1 functions

$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots$

in $\Delta$, where we let $a_1 = 1$. 