

Math 506 Midterm

Mar. 2, 2010

- (1) Show that $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\Delta^* = \{0 < |z| < 1\}$ are not conformally equivalent, i.e., there does not exist a holomorphic map $f : \mathbb{C}^* \rightarrow \Delta^*$ which is 1-1 and onto.
- (2) Compute the integral

$$\int_0^{\infty} \sin(x^2) dx$$

You may assume the well-known integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

- (3) Suppose that f and g are holomorphic in an open set containing the disc $|z| \leq 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Let

$$f_t(z) = f(z) + tg(z)$$

Show that if t is sufficiently small, then

- (a) $f_t(z)$ has a unique zero in $|z| \leq 1$, and
- (b) if $\xi(t)$ is the zero, it is a continuous function in t .
- (4) Let $f(z)$ be an entire function satisfying $|f(z_1 + z_2)| \leq |f(z_1)| + |f(z_2)|$ for all $z_1, z_2 \in \mathbb{C}$. Then $f(z) \equiv az + b$ for some constants $a, b \in \mathbb{C}$.