Math 506 Homework Solution

- Let $f(x,y) \in \mathcal{O}_2[[x,y]]$. Suppose that

$$f(x,y) = (x - c_1 y)(x - c_2 y)...(x - c_m y) + \sum_{j+k>m} a_{jk} x^j y^k \tag{0.1}$$

where $c_1, c_2, ..., c_m, a_{jk} \in \mathbb{C}$ are constants. We call m the multiplicity of $f(x,y)$ at the origin. Show that if $c_1, c_2, ..., c_m$ are m distinct numbers, then there exists $g_1(y), g_2(y), ..., g_m(y) \in \mathcal{O}_1[[y]]$ such that

$$f(x,y) = (x - g_1(y))(x - g_2(y))...(x - g_m(y))h(x,y)$$

where $h(x,y) \in \mathcal{O}_2[[x,y]]$ and $h(0,0) \neq 0$.

Proof. By Weierstrass preparation theorem,

$$f(x,y) = p(x,y)h(x,y)$$

where $p(x,y) = x^m + a_1(y)x^{m-1} + ... + a_m(y) \in \mathcal{O}_2[[y]][x]$ is a Weierstrass polynomial and $h(0,0) \neq 0$. Since the algebraic closure of the quotient field $\mathcal{O}_2([y])$ of $\mathcal{O}_2[[y]]$ is generated by $\sqrt[n]{y}$ for all $n \in \mathbb{Z}^+$, we can factorize $p(x,y)$ into a product of monomials:

$$p(x,y) = (x - g_1(y))(x - g_2(y))...(x - g_m(y))$$

where $g_k(y) \in \mathcal{O}_2([\sqrt[n]{y}])$ for some positive integer n. Alternatively, there is a cyclic field extension $\mathcal{O}_2([y]) \to \mathcal{O}_2([t])$ sending y to t^n such that

$$p(x, t^n) = (x - b_1(t))(x - b_2(t))...(x - b_m(t))$$

where $b_k(t) = g_k(t^n) \in \mathcal{O}_2([t])$.

By comparing $p(x,t^n)$ and $f(x,t^n)$, we conclude that

$$b_k(t) = c_k t^n + \sum_{l>n} b_{kl} t^l$$

Let $G = \mathrm{Gal}(\mathcal{O}_2([t])/\mathcal{O}_2([y])) = \mathbb{Z}/n\mathbb{Z}$ be the Galois group of the field extension $\mathcal{O}_2([y]) \to \mathcal{O}_2([t])$. Then G acts on $\mathcal{O}_2([t])$ by sending t to ξt, where $\xi^n = 1$. And G acts as a subgroup of the symmetric group S_m on $b_1(t), b_2(t), ..., b_m(t)$: it sends each $b_k(t)$ to $b_l(t)$. That is, for each $\xi^n = 1$ and each $1 \leq k \leq m$, there exists $1 \leq l \leq m$ such that

$$b_k(\xi t) = b_l(t)$$

Since c_k are distinct, we must have $k = l$ for all $\xi^n = 1$ and k. Therefore,

$$b_k(\xi t) = b_k(t)$$

for all $\xi^n = 1$. It follows that $b_k(t)$ is a power series in t with only terms in t^{jn} for $j \in \mathbb{Z}$. It follows that $g_k(y) \in \mathcal{O}_2[[y]]$ and we are done. □

- Let $f \in \mathcal{O}_2[[z,w]]$ where $f \not\equiv 0$, f is irreducible and $f(0,0) = 0$. Then $\mathcal{O}_2[[z,w]]/(f) \cong \mathcal{O}_1$ if and only if $\mathcal{O}_2[[z,w]]/(f)$ is integrally closed.

  Proof. If $\mathcal{O}_2[[z,w]]/(f) \cong \mathcal{O}_1$, then $\mathcal{O}_2[[z,w]]/(f)$ is, of course, integrally closed.

  Suppose that $\mathcal{O}_2[[z,w]]/(f)$ is integrally closed. If $f_z(0,0) \neq 0$ or $f_w(0,0) \neq 0$, then by inverse function theorem, $\mathcal{O}_2[[z,w]]/(f) \cong \mathcal{O}_1$.

  Suppose that $f_z(0,0) = f_w(0,0) = 0$. We write

  $$f(z,w) = \sum_{j=0}^{m} b_j z^j w^{m-j} + \sum_{k+l>m} a_{kl} z^k w^l$$

  where $m \geq 2$ and $b_0, b_1, ..., b_m \in \mathbb{C}$ are not all zero. After a linear transformation sending $(z,w) \to (z, w + \lambda z)$ for some λ, we may assume that $f(z,0) \neq 0$, i.e., $b_m \neq 0$. By Weierstrass preparation theorem, we have

  $$f(z,w) = (z^m + a_1(w) z^{m-1} + ... + a_m(w)) h(z,w)$$

  where $h(0,0) \neq 0$. Obviously, $a_k(w)$ has a zero at $w = 0$ of multiplicity at least k for each $1 \leq k \leq m$. We write $a_k(w) = w^k b_k(w)$ for some $b_k(w) \in \mathcal{O}_2[[w]]$. Then

  $$z^m + a_1(w) z^{m-1} + ... + a_m(w) = w^m \left(\left(\frac{z}{w} \right)^m + b_1(w) \left(\frac{z}{w} \right)^{m-1} + ... + b_m(w) \right)$$

  It follows that $z/w \in \mathcal{O}_2[[z,w]]/(f)$ since $\mathcal{O}_2[[z,w]]/(f)$ is integrally closed. Therefore,

  $$\frac{z}{w} = g(z,w) \Rightarrow z - wg(z,w) = 0$$

  in $\mathcal{O}_2[[z,w]]/(f)$ for some $g(z,w) \in \mathcal{O}_2[[z,w]]$. Thus $z - wg(z,w) \in (f)$. And since f and $z - wg(z,w)$ are irreducible, we must have $z - wg(z,w) = c(z,w) f$ for some $c(0,0) \neq 0$. Then we have $f_z(0,0) \neq 0$. Contradiction. $\square$

- Let f be a holomorphic function on $\mathbb{C}^n \backslash \{(z_1, z_2, ..., z_n) : z_1 = z_2 = 0\}$. Show that f can be extended to a holomorphic function on $\mathbb{C}^n$.

  Proof. We define

  $$g(z_1, z_2, ..., z_n) = \int_{|w|=1} \frac{f(w, z_2, ..., z_n)}{w - z_1} dz_1$$

  Clearly, $g(z_1, z_2, ..., z_n)$ is holomorphic in $|z_1| < 1$ and $g(z_1, z_2, ..., z_n) = f(z_1, z_2, ..., z_n)$ in $\{|z_1| < 1, z_2 \neq 0\}$. Therefore, $f(z_1, z_2, ..., z_n)$ can be extended to a holomorphic function on $\mathbb{C}^n$. $\square$