Math 506 Homework 5

(1) Let $f : \mathbb{C} \to \mathbb{C}/\Lambda$ be a nonconstant holomorphic map from $\mathbb{C}$ to a complex torus $\mathbb{C}/\Lambda$. Show that f is surjective.

(2) We know that the upper half plane $H = \{\operatorname{Im} z > 0\}$ and the unit disk $\Delta = \{|z| < 1\}$ are conformally equivalent, i.e., biholomorphic. Is this true in higher dimension? That is, take $H_n = \{(z_1, z_2, ..., z_n) : \operatorname{Im} z_1 > 0\}$ and $\Delta^n = \{(z_1, z_2, ..., z_n) : |z_1| < 1, |z_2| < 1, ..., |z_n| < 1\}$ in $\mathbb{C}^n$. Are H_n and Δ^n conformally equivalent?

(3) Let f be a holomorphic function on $\mathbb{C}^n$. If there are positive constants C and λ such that

$$|f(z_1, z_2, ..., z_n)| \leq C(|z_1|^\lambda + |z_2|^\lambda + ... + |z_n|^\lambda)$$

for all $(z_1, z_2, ..., z_n) \in \mathbb{C}^n$, then $f(z_1, z_2, ..., z_n)$ is a polynomial in $z_1, z_2, ..., z_n$ of degree $\leq \lambda$.

(4) Let f be a holomorphic function on $\mathbb{C}^n$. If $f(\mathbb{C}^n)$ misses at least two values, then f must be a constant.

(5) Let D be an open set in $\mathbb{C}^n$ and let $\{f_k\}$ be a sequence of holomorphic functions on D. If f_k converges to a function f uniformly on D, then f is holomorphic on D and

$$\lim_{k\to\infty} \frac{\partial f_k}{\partial z_l} = \frac{\partial f}{\partial z_l}$$

on D for $l = 1, 2, ..., n$.

(6) Let f be a holomorphic function on $\mathbb{C}^n \backslash \{(z_1, z_2, ..., z_n) : z_1 = z_2 = 0\}$. Show that f can be extended to a holomorphic function on $\mathbb{C}^n$.