Math 506 Homework 5

(1) Let \( f: \mathbb{C} \to \mathbb{C}/\Lambda \) be a nonconstant holomorphic map from \( \mathbb{C} \) to a complex torus \( \mathbb{C}/\Lambda \). Show that \( f \) is surjective.

(2) We know that the upper half plane \( H = \{ \text{Im } z > 0 \} \) and the unit disk \( \Delta = \{ |z| < 1 \} \) are conformally equivalent, i.e., biholomorphic. Is this true in higher dimension? That is, take \( H_n = \{ \text{Im } z_1 > 0 \} \) and \( \Delta^n = \{ |z_1| < 1, |z_2| < 1, \ldots, |z_n| < 1 \} \) in \( \mathbb{C}^n \). Are \( H_n \) and \( \Delta^n \) conformally equivalent?

(3) Let \( f \) be a holomorphic function on \( \mathbb{C}^n \). If there are positive constants \( C \) and \( \lambda \) such that

\[
|f(z_1, z_2, \ldots, z_n)| \leq C(|z_1|^\lambda + |z_2|^\lambda + \ldots + |z_n|^\lambda)
\]

for all \( (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \), then \( f(z_1, z_2, \ldots, z_n) \) is a polynomial in \( z_1, z_2, \ldots, z_n \) of degree \( \leq \lambda \).

(4) Let \( f \) be a holomorphic function on \( \mathbb{C}^n \). If \( f(\mathbb{C}^n) \) misses at least two values, then \( f \) must be a constant.

(5) Let \( D \) be an open set in \( \mathbb{C}^n \) and let \( \{ f_k \} \) be a sequence of holomorphic functions on \( D \). If \( f_k \) converges to a function \( f \) uniformly on \( D \), then \( f \) is holomorphic on \( D \) and

\[
\lim_{k \to \infty} \frac{\partial f_k}{\partial z_l} = \frac{\partial f}{\partial z_l}
\]

on \( D \) for \( l = 1, 2, \ldots, n \).

(6) Let \( f \) be a holomorphic function on \( \mathbb{C}^n \setminus \{(z_1, z_2, \ldots, z_n): z_1 = z_2 = 0\} \). Show that \( f \) can be extended to a holomorphic function on \( \mathbb{C}^n \).