Math 506 Homework 4

We use the notation $\mathcal{O}_n[[z_1, z_2, \ldots, z_n]]$ to denote the ring of germs of analytic functions in $n$ variables $z_1, z_2, \ldots, z_n$.

(1) Let $f(x, y) \in \mathcal{O}_2[[x, y]]$. Suppose that

$$f(x, y) = (x - c_1 y)(x - c_2 y)\ldots(x - c_m y) + \sum_{j+k>m} a_{jk} x^j y^k$$

where $c_1, c_2, \ldots, c_m, a_{jk} \in \mathbb{C}$ are constants. We call $m$ the multiplicity of $f(x, y)$ at the origin. Show that if $c_1, c_2, \ldots, c_m$ are $m$ distinct numbers, then there exists $g_1(y), g_2(y), \ldots, g_m(y) \in \mathcal{O}_1[[y]]$ such that

$$f(x, y) = (x - g_1(y))(x - g_2(y))\ldots(x - g_m(y))h(x, y)$$

where $h(x, y) \in \mathcal{O}_2[[x, y]]$ and $h(0, 0) \neq 0$.

(2) Let $f(x, y) \in \mathcal{O}_2[[x, y]]$. Suppose that $f(x, y)$ has multiplicity 2 at the origin, i.e., $f(x, y)$ is given by (0.1) with $m = 2$. Then there exists an automorphism $\phi : \mathcal{O}_2[[x, y]] \to \mathcal{O}_2[[x, y]]$ such that $\phi(f(x, y)) = x^2$ or $\phi(f(x, y)) = x^2 - y^n$ for some $n \geq 2$.

(3) Let $\mathcal{O}_1[[z]]$ be the quotient field of $\mathcal{O}_1[[z]]$. Then any finite extension of $\mathcal{O}_1[[z]]$ is itself and given by $\phi : \mathcal{O}_1[[z]] \to \mathcal{O}_1[[z]]$ with $\phi(z) = z^n$ for some $n > 0$.

(4) Let $\mathbb{C}[[z, w]]$ be the ring of formal power series. Prove Weierstrass Preparation Theorem on $\mathbb{C}[[z, w]]$. That is, let

$$f(z, w) = w^m + \sum_{(i, j) \neq (0, m)} a_{ij} z^i w^j \in \mathbb{C}[[z, w]]$$

Then $f(z, w) = g(z, w)h(z, w)$ where $g(z, w) = w^d + a_1(z)w^{d-1} + \ldots + a_d(z)$ is a Weierstrass polynomial and $h(0, 0) \neq 0$.

(5) Let $f \in \mathcal{O}_2[[z, w]]$ where $f \neq 0$, $f$ is irreducible and $f(0, 0) = 0$. Then $\mathcal{O}_2[[z, w]]/(f) \cong \mathcal{O}_1$ if and only if $\mathcal{O}_2[[z, w]]/(f)$ is integrally closed.