Math 506 Homework 2 Solution

(7) Let $\xi(s) = \sum_{n=1}^{\infty} n^{-s}$ be the Riemann zeta function. Show that

$$|\xi(s)| \geq \frac{6}{\pi^2}$$

for $\mathrm{Re}(s) \geq 2$.

Proof. We have

$$\begin{aligned} |\xi(s)| &= \left| \frac{1}{1-2^{-s}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} \right| \\ &\geq \frac{1}{1+2^{-2}} \left(1 - \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \right) \end{aligned}$$

for $\mathrm{Re}(s) \geq 2$. Note that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Therefore,

$$|\xi(s)| \geq \frac{4}{5}\left(2 - \frac{\pi^2}{8}\right) > \frac{6}{\pi^2}$$

for $\mathrm{Re}(s) \geq 2$. □