Math 506 Homework 2

(1) Compute
$$\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$$
(2) Let M and N be two topological spaces. Two continuous maps $f : M \to N$ and $g : M \to N$ are called homotopic, written as $f \sim g$, if there exists a continuous map $F : M \times [0,1] \to N$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. And M and N are homotopic to each other if there exists two continuous maps $f : M \to N$ and $g : N \to M$ such that $f \circ g \sim 1_N$ and $g \circ f \sim 1_M$, where 1_M and 1_N are identity maps on M and N, respectively. Show that $M = \mathbb{R}^2 \backslash \{\text{two points}\}$ and $N = \{x^2 + y^2 = 1\} \cup \{(x-2)^2 + y^2 = 1\} \subset \mathbb{R}^2$ are homotopic.
(3) Find the Laurent Series of $f(z) = (z-1)^{-2}(z-2)^{-1}$ in
   (a) $|z| < 1$;
   (b) $1 < |z| < 2$;
   (c) $|z| > 2$.
(4) Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \operatorname{Re} z > 0\}$.
(5) In the above problem, futher show that the solution must be real. What happens to the solution as $\lambda \to 1$?
(6) Let γ be the rectangular path along
$$\left[n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, \right.$$
$$\left. -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni, n + \frac{1}{2} + ni \right]$$
and evaluate the integral
$$\int_\gamma \frac{\cot(\pi z)}{(z+a)^2} dz$$
for $a \notin \mathbb{Z}$. Use your result to deduce that
$$\frac{\pi^2}{\sin^2(\pi a)} = \sum_{n=-\infty}^{n} \frac{1}{(a+n)^2}$$
(7) Let $\xi(s) = \sum_{n=1}^\infty n^{-s}$ be the Riemann zeta function. Show that
$$|\xi(s)| \geq \frac{6}{\pi^2}$$
for $\operatorname{Re}(s) \geq 2$.