

Math 506 Homework 1 Solution

- (2) Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \dots, (f(n), g(n))$ in \mathbb{R}^2 are the vertices of a regular n -gon in counter-clockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n - 1$.

Proof. Let $p \in \mathbb{C}$ be the center of the regular n -gon. Then

$$\frac{f(k+1) + ig(k+1) - p}{f(k) + ig(k) - p} = \alpha = e^{2\pi i/n}$$

for $1 \leq k \leq n - 1$. Let $F(z) = f(z) + ig(z) - p$ and $G(z) = F(z+1) - \alpha F(z)$. Then $G(z)$ has zeroes at $z = 1, 2, \dots, n - 1$. Consequently, $\deg G(z) \geq n - 1$ and hence $\deg F(z) \geq n - 1$. It follows that $\max(\deg f, \deg g) \geq n - 1$. \square

- (4) Compute the integrals

$$\int_0^\infty \sin(x^2) dx \text{ and } \int_0^\infty \cos(x^2) dx$$

Solution. Obviously,

$$\int_0^\infty e^{ix^2} dx = \int_0^\infty \cos(x^2) dx + i \int_0^\infty \sin(x^2) dx$$

Consider the contour integral

$$\int_{C_R} e^{iz^2} dz$$

along the boundary of the sector $\{|z| < R, 0 < \arg z < \pi/4\}$, oriented counter-clockwise. Then

$$0 = \int_{C_R} e^{iz^2} dz = \int_0^R e^{iz^2} dz + \int_{A_R} e^{iz^2} dz - e^{\pi i/4} \int_0^R e^{-z^2} dz$$

where A_R is the arc $\{Re^{i\theta} : 0 < \theta < \pi/4\}$. Since

$$\begin{aligned} \left| \int_{A_R} e^{iz^2} dz \right| &\leq R \int_0^{\pi/4} \exp(-R^2 \sin(2\theta)) d\theta \\ &\leq R \int_0^{\pi/4} \exp(-4R^2\theta/\pi) d\theta \end{aligned}$$

it follows that

$$\lim_{R \rightarrow \infty} \int_{A_R} e^{iz^2} dz = 0$$

Therefore,

$$\int_0^\infty e^{iz^2} dz = e^{\pi i/4} \int_0^\infty e^{-x^2} dx = \frac{\sqrt{2\pi}}{4} + i \frac{\sqrt{2\pi}}{4}$$

Hence

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

- (5) Suppose that $f : D = \{|z| < 1\} \rightarrow \mathbb{C}$ is holomorphic. Show that the diameter

$$d = \sup_{z, w \in D} |f(z) - f(w)|$$

of the image of f satisfies

$$2|f'(0)| \leq d$$

and the equality holds if and only if $f(z) = a + bz$ is linear.

Proof. Let us consider $g(z) = z^{-1}(f(z) - f(-z))$. Clearly, $g(z)$ extends to an analytic function on D and $g'(0) = 2f'(0)$. By maximum modulus,

$$|g'(0)| = 2|f'(0)| \leq \limsup_{|z| \rightarrow 1} \frac{f(z) - f(-z)}{z} \leq d$$

For the second part of the problem, I can prove it under the additional assumption that $f(z)$ is analytic along the circle $\partial D = \{|z| = 1\}$. I am still on the look for a proof of the second part in general.

Suppose that $f(z)$ is also analytic along ∂D . That is, $f(z)$ is analytic in $|z| < R$ for some $R > 1$. When the quality holds, we necessarily have $g(z) = 2f'(0)$ for all z by maximum modulus. WLOG, we assume that $d = 2f'(0) = 2$. Then $f(z) - f(-z) = 2z$ for all $|z| \leq 1$.

Fixing a point z with $|z| = 1$, we have

$$\begin{aligned} |f(w) - f(-z)| &\leq 2 \Rightarrow \left| \frac{f(w) - f(z)}{z} + 2 \right| \leq 2 \\ (0.1) \quad &\Rightarrow \operatorname{Re} \left(\frac{f(w) - f(z)}{z} \right) \leq 0 \\ &\Rightarrow \operatorname{Re} \left(\frac{f(w) - f(z)}{w - z} \frac{w - z}{|w - z|z} \right) \leq 0 \end{aligned}$$

for all $|w| \leq 1$ and $w \neq z$.

If we let $w \rightarrow z$ and $|w| = 1$, we see that

$$\lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z} = f'(z) \quad \text{and} \quad \lim_{\substack{w \rightarrow z \\ |w|=1}} \frac{w - z}{|w - z|z} = \pm i$$

It follows that $\operatorname{Im}(f'(z)) = 0$ for all $|z| = 1$. By applying maximum modulus to $\exp(if'(z))$, we see that $f'(z) \equiv 1$ by noticing that $f'(0) = 1$. Therefore, $f(z) = a + z$ for some constant a . \square

- (8) Morera's theorem states that if f is continuous in an open set $D \subset \mathbb{C}$ and $\int_T f(z)dz = 0$ for all triangles $T \subset D$, then f is analytic in D . Does the theorem still hold if we replace triangles by rectangles? What if we replace triangles by circles?

Proof. We will prove it for circles. The case of rectangles follows from a similar argument.

First we prove it with the additional assumption that $f(z) \in C^2(D)$. At a point $z_0 \in D$, we have

$$|f(z) - f(z_0) - f_z(z_0)(z - z_0) - f_{\bar{z}}(z_0)(\bar{z} - \bar{z}_0)| \leq C|z - z_0|^2$$

in some open neighborhood of z_0 and for some constant C , where $f_z = \partial f / \partial z$ and $f_{\bar{z}} = \partial f / \partial \bar{z}$. Therefore,

$$\begin{aligned} \left| \int_{|z-z_0|=r} f(z)dz \right| &\geq \left| \int_{|z-z_0|=r} f_{\bar{z}}(z_0)(\bar{z} - \bar{z}_0)dz \right| - 2\pi Cr^3 \\ &= 2\pi |f_{\bar{z}}(z_0)|r^2 - 2\pi Cr^3 \end{aligned}$$

for r sufficiently small. It follows that $f_{\bar{z}}(z_0) = 0$ and hence f is analytic in D .

Indeed, we have proved the following: If $f(z) \in C^2(D)$ and for every point $z_0 \in D$, there exists $\varepsilon > 0$ such that

$$(0.2) \quad \int_{|z-z_0|=r} f(z)dz = 0$$

for all $r < \varepsilon$, then $f(z)$ is analytic in D .

Now let us deal with the general case. The problem is local. So it suffices to prove it for $D = \{|z| < 1\}$. Indeed, it suffices to prove that $f(z)$ is analytic at 0.

We may further assume that $f(z)$ is continuous on $\{|z| \leq 1\}$. So we can extend $f(z)$ to a bounded continuous function on \mathbb{C} . We fix a number $0 < \alpha < 1/2$. Let $g(z) \in C^\infty(\mathbb{C})$ be a C^∞ function with compact support contained in $\{|z| < 1 - 2\alpha\}$. We define

$$(0.3) \quad F(z) = f * g = \int_{\mathbb{C}} f(z-w)g(w)dw \wedge d\bar{w}$$

Obviously, $F \in C^\infty(\mathbb{C})$.

We claim that $F(z)$ has the property (0.2) for all $|z_0| < \alpha$.

Fix z_0 such that $|z_0| < \alpha$. Then for all $r < \alpha$,

$$\begin{aligned}
& \int_{|z-z_0|=r} F(z) dz \\
&= \int_{|w-z_0|<1-r} \left(\int_{|z-z_0|=r} f(z-w) dz \right) g(w) \wedge dw \wedge d\bar{w} \\
&\quad + \int_{|w-z_0|\geq 1-r} \left(\int_{|z-z_0|=r} f(z-w) dz \right) g(w) dw \wedge d\bar{w} \\
&= \int_{|w-z_0|<1-r} \left(\int_{|z-(z_0-w)|=r} f(z) dz \right) g(w) \wedge dw \wedge d\bar{w} = 0
\end{aligned}$$

So we may conclude that $F(z)$ is analytic in $\{|z| < \alpha\}$.

We may regard g as a bounded functional on $C(\mathbb{C})$ defined by

$$\varphi_g(h) = \int_{\mathbb{C}} h(w) g(w) dw \wedge d\bar{w}$$

for all $h \in C(\mathbb{C})$.

There exists a sequence of $g_n \in C^\infty(\mathbb{C})$ with

$$\text{supp}(g_n) \subset \{|z| < 1 - 2\alpha\}$$

such that the corresponding functionals φ_n converges to the delta function at 0, i.e.,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{C}} h(w) g_n(w) dw = h(0)$$

for all $h \in C(\mathbb{C})$. In addition, we may choose g_n such that

$$(0.4) \quad \int_{\mathbb{C}} |g_n| dx dy \leq M$$

for some constant M .

Let $F_n = f * g_n$ as defined in (0.3). Then $F_n(z)$ is analytic in $\{|z| < \alpha\}$ and

$$\lim_{n \rightarrow \infty} F_n(z) = f(z)$$

for all z . In addition, F_n is uniformly bounded by (0.4). It follows that $f(z)$ is analytic in $\{|z| < \alpha\}$. \square